Lecture 23: **03/13/09**

- Formalism III
  - Eigenfunctions of a hermitian operator
  - statistical interpretation
  - Dirac notation

III_3 Operators and Observables:

Recap

- Observables are represented by hermitian operators.

$$\langle Q \rangle = \langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle$$

also $\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle$ for all $f(x), g(x)$

III_4 Eigenfunctions of a hermitian operator:

- Determinate states are eigenfunctions of the hermitian operator $\hat{Q}$.

$$\hat{Q} \Psi = q \Psi \quad \} \text{ eigenvalue equation}$$

For discrete spectra:
- Eigenvalues q are real
- eigenfunctions are orthonormal: $\langle f_n | f_m \rangle = \delta_{nm}$
- eigenfunctions are complete

$\Rightarrow$ Can expand any wave function in terms of the (base) functions:

- For degenerated spectrum ($q' = q$)

  ⇒ can construct orthonormal eigenfunctions within each degenerated subspace

  ⇒ eigenfunctions can be <u>chosen</u> to be orthogonal

C) • <u>Axiom</u>: The set of eigenfunctions of an observable operator $\hat{Q}$ is <u>complete</u>.

⇒ Any function in Hilbert space can be expressed as a linear combination of the eigenfunctions!

Case II Continuous spectra:

$\Rightarrow$ Eigenfunctions can not be normalized

$\Rightarrow$ But still: the three essential properties (reality, orthogonality, and completeness) hold

$\Rightarrow$ $\sum \rightarrow \int dx$; $\underbrace{\delta_{nm}}_{\text{Kronecker delta}} \rightarrow \delta(\quad)$ Dirac-delta function

Example: Position operator $\hat{x} = x$

eigenvalue equation: $x \, g_y(x) = y \, g_y(x)$ eigenfunction; eigenvalue (fixed number)

$\Rightarrow$ Solutions: eigenfunctions: $g_y(x) = A \, \delta(x-y)$

Dirac delta functions, zero except for one point $x = y$

$\Rightarrow$ these functions are not square-integrable

$\Rightarrow$ do not represent a physical particle

$\Rightarrow$ can not be normalized

$\Rightarrow$ but still: eigenvalues are real (property a)

The Dirac delta function:

- Kronecker delta : $\delta_{ik} = \begin{cases} 0, & \text{if } i \neq k \\ 1, & \text{if } i = k \end{cases}$ $\Rightarrow f_k = \sum_i f_i \delta_{ik}$

select f_k

- Similar: for continuous variable:

$$\boxed{f(y) = \int_{-\infty}^{+\infty} f(x)\, \delta(x-y)\, dx} \leftarrow \text{select } f(\text{point } y)$$

with the Dirac delta function: $\delta(x-y) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \frac{\varepsilon}{(x-y)^2 + \varepsilon^2}$

- Some useful equations:

$$\delta(ax) = \frac{1}{|a|} \delta(x) \qquad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

$$\int_{-\infty}^{+\infty} \delta(x-y)\, \delta(y-x')\, dy = \delta(x-x')$$

$$\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x')}\, dk$$

$\Rightarrow$ can recover a kind of "ersatz orthonormality"

$$\langle g_{y'}(x) | g_y(x) \rangle = \int_{-\infty}^{+\infty} g_{y'}^*(x)\, g_y(x)\, dx$$

$$= |A|^2 \int_{-\infty}^{+\infty} \delta(x-y')\, \delta(x-\cancel{y})\, dx = |A|^2\, \delta(y-y')$$

$\Rightarrow$ if we pick $A=1$

$\Rightarrow \quad g_y(x) = \delta(x-y)$ } eigenfunction of $\hat{x}$ with real eigenvalue

$\Rightarrow \quad \langle g_{y'} | g_y \rangle = \delta(\cancel{y}-y')$ } "Dirac orthonormal" (property 6)

Dirac delta function

Also: Set of eigenfunctions of $\hat{x}$ is complete:

(property c)

$$f(x) = \int_{-\infty}^{+\infty} c(y)\, g_y(x)\, dy = \int_{-\infty}^{+\infty} c(y)\, \delta(x-y)\, dy = c(x)$$

$\uparrow$ $\sum_n$ $\qquad$ $\uparrow$ replaces c_n

with $c(y) = f(y) = \langle g_y \mid f(x) \rangle$: project. amplit.

$$\Rightarrow \quad \Psi(y,t) = \langle g_y \mid \Psi \rangle$$

} wave function = projection amplitude into position space

III_5 Generalized Statistical Interpretation:

Found that set of eigen functions $\{f_n\}$ of an operator representing an observable Q is orthonormal and complete.

$\Rightarrow$ can expand any wave function in terms of these base functions:

Note: different operator $\hat{Q}'$ $\Rightarrow$ different set of $\{f_n\}'$

$$\Psi(x,t) = \sum_n c_n(t)\, f_n(x) \quad \Big\} \text{ discrete spectrum}$$

$$= \int c(q,t)\, f_q(x)\, dq \quad \Big\} \text{ continuous spectrum}$$

with: $\underbrace{c_n(t)} = \langle f_n | \Psi(x,t) \rangle =$ projection amplitude; quantum ampl.

Note: coefficients include the time dependence!

c_n: tells you "how much f_n is contained in $\Psi(x,t)$"

- Statistical Interpretation:

If one measures an observable Q on a particle in the state $\Psi(x,t)$, one is certain to get one of the eigenvalues of the corresponding hermitian operator $\hat{Q}$!

If: I: spectrum of $\hat{Q}$ is discrete:

Probability of getting the eigenvalue q_n associated with the eigenfunction $f_n(x)$ is $|c_n|^2$, where $c_n = \langle f_n | \Psi \rangle$

II: continuous spectrum:

$\Rightarrow$ eigenvalues q, eigenfunctions $f_q(x)$

$\Rightarrow$ probability of getting a result in the interval $[q, q+dq]$ is $|c(q)|^2 dq$, where $c(q) = \langle f_q | \Psi \rangle$

Note: Upon measurement, the wave function "collapses" to the corresponding eigenstate!

- Total probability of getting a result = 1

$$\Rightarrow \sum_n |c_n|^2 = 1$$

Proof:

$$\underline{1} = \langle \Psi | \Psi \rangle = \langle \sum_{n'} c_{n'} f_{n'} | \sum_n c_n f_n \rangle$$

$$= \sum_{n'} \sum_n c_{n'}^* c_n \underbrace{\langle f_{n'} | f_n \rangle}_{\delta_{n'n}} = \underline{\sum_n |c_n|^2}$$

- Expectation value:

recall: for energy $\langle H \rangle = \langle E \rangle = \sum_n E_n |c_n|^2$

$$\langle Q \rangle = \sum_n q_n |c_n|^2$$

q_n: possible outcome of measurement

$|c_n|^2$: probability of getting that result

example: for position:

$$\langle x \rangle = \int_{-\infty}^{+\infty} y \, |c(y)|^2 \, dy \quad \text{with } c(y) = \langle g_y | \Psi \rangle = \int_{-\infty}^{+\infty} \delta(x-y) \Psi(x,t) \, dx = \Psi(y,t)$$

$$\Rightarrow |c(y)|^2 = |\Psi(y,t)|^2$$

$|c(y)|^2$: prob. of measuring value y as position

III_6 Dirac Notation:

Dirac: chop bracket notation for inner product in two pieces:

bra : $\langle \alpha |$

ket : $| \beta \rangle$

- ket $| \beta \rangle$

"all that can be known"

  - represents the state of the system/particle
  - in position space: $| \beta \rangle$ is represented by a function $\Psi(x,t)$
  - in vector space: state $| \beta \rangle$ is represented by a state vector: $| \beta \rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$