Lecture 31: **04/08/09**

- Free particle II: particle dynamics
- Energy-time uncertainty principle

IV$_4$ The Generalized Uncertainty Principle:

Recap

$$\boxed{\sigma_A^2 \sigma_B^2 \underset{\text{always}}{\geq} \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle\right)^2}$$

where : $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$: Commutator of $\hat{A}$ and $\hat{B}$

Example: $A = \hat{x}$ $\quad \hat{B} = \hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$ $\Rightarrow$ $\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$

Heisenberg uncertainty principle

IV$_5$ The Free Particle II: Particle Dynamics:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p)\, e^{ipx/\hbar} \underbrace{e^{-i\frac{p^2}{2m}\frac{t}{\hbar}}}_{\text{time dependence of state of definite momentum}} dp$$

have: $\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p)\, e^{ipx/\hbar}\, e^{-i\frac{p^2}{2m}t/\hbar}\, dp$

for free particle

<u>will see that:</u>

1) If $\Phi(p)$ has some narrow distribution around some value p_0 $\Rightarrow$ wave packet will move at the <u>group velocity</u>:

$$V_g = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{d}{dp}\left(\frac{p^2}{2m}\right) = \frac{p_0}{m} = \frac{\hbar k}{m} = V_{classical}$$

2) Width of the wave packet in space (!) will change – "dispersion"

reason:

phase velocity $= v_p = \frac{\omega}{k} = \frac{E}{p} = \frac{p}{2m} = \frac{\hbar k}{2m}$

$=$ velocity of individual states of definite momenta

$\Rightarrow$ depends on p (or k)

$\Rightarrow$ relative phase between state of definite momentum in the superposition will change with time

$\Rightarrow$ width of wave packet in position space changes

→ Example: Gaussian wave packet:

$$\Phi(p,t) = \frac{1}{(2\pi)^{1/4}\sqrt{\sigma_p}}\, e^{-p^2/4\sigma_p^2}\, \underbrace{e^{-i\frac{p^2}{2m}t/\hbar}}_{\text{now with time dependence!}}$$

this gives:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{1/4}\sqrt{\sigma_p}}\, e^{-p^2/4\sigma_p^2}\, e^{-i\frac{p^2}{2m}t/\hbar}\, e^{i\frac{px}{\hbar}}\, dp$$

= .... some math ...

$$= \frac{1}{(2\pi)^{1/4}\sqrt{\frac{\hbar}{2\sigma_p}}} \frac{1}{\sqrt{1+i\frac{2\sigma_p^2}{m\hbar}t}} \exp\left\{-\frac{x^2\sigma_p^2}{\hbar^2}\frac{1}{\left(1+i\frac{2\sigma_p^2}{m\hbar}t\right)}\right\}$$

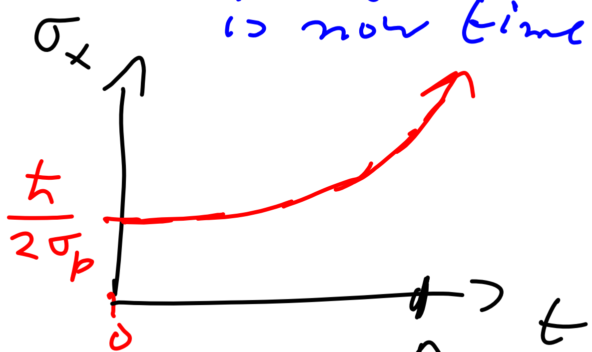
=) probability density:

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\left(\frac{\hbar}{2\sigma_p}\right)}\,\frac{1}{\sqrt{1+\left(\frac{2\sigma_p^2}{m\hbar}\cdot t\right)^2}}\,\exp\left\{\underbrace{\frac{-2x^2\sigma_p^2}{\hbar^2\left(1+\left(\frac{2\sigma_p^2}{m\hbar}t\right)^2\right)}}_{|\Psi(x,t)|^2 \propto \exp\left(-\frac{2x^2}{4\sigma_x(t)^2}\right)}\right\}$$

=) still Gaussian, but the width is now time dependent:

$$\sigma_x = \frac{\hbar}{2\sigma_p}\left(1+\left(\frac{2\sigma_p^2}{m\hbar}t\right)^2\right)^{1/2}$$

=) wave packet broadens!



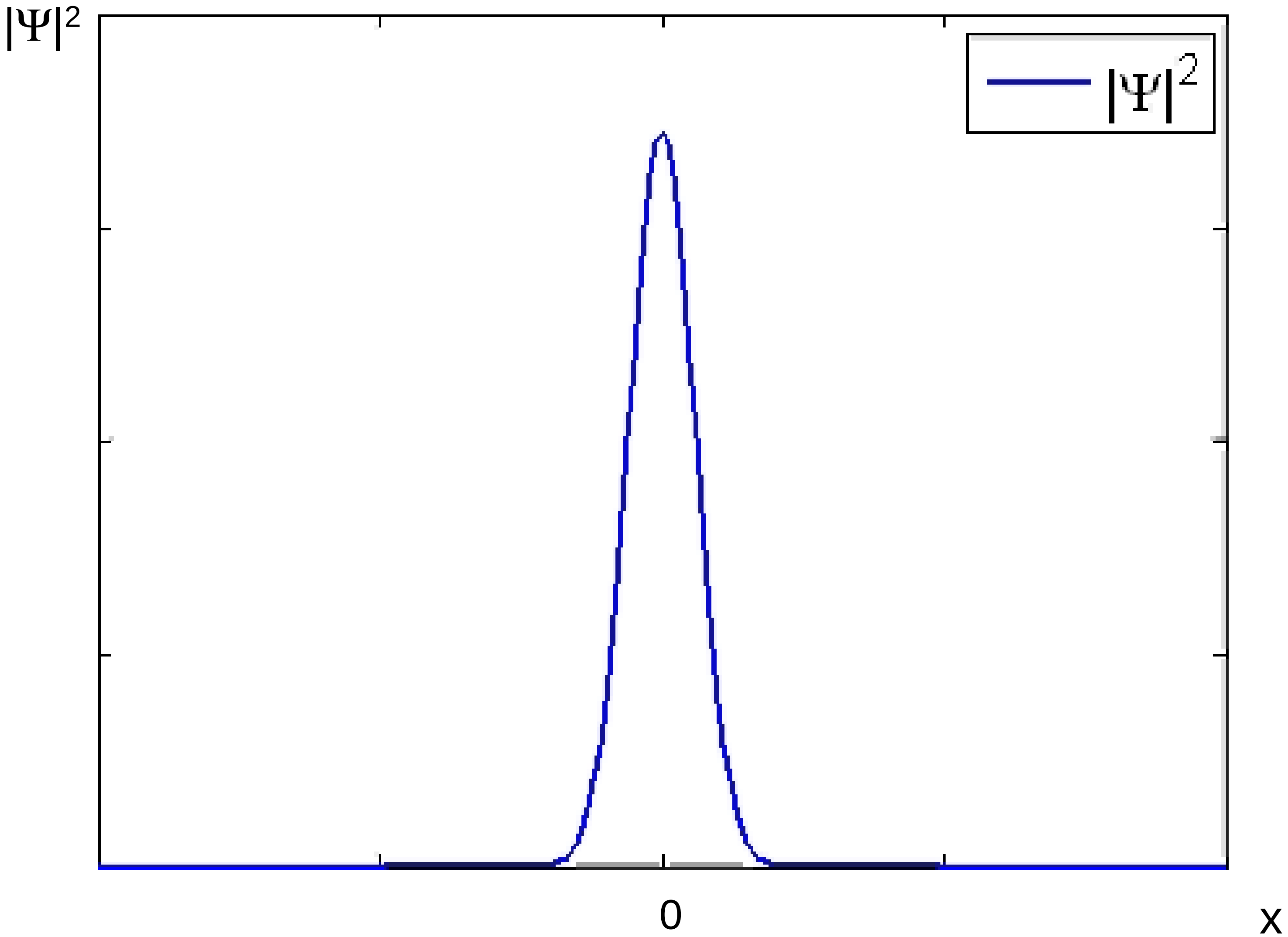
time scale of broadening $\approx \frac{m\hbar}{2\sigma_p^2} \rightarrow \frac{2\sigma_p^2}{m\hbar}t = 1$

– spread in phase velocities of states of definite momentum: $\Delta V_p = \frac{\hbar\Delta k}{2m} = \frac{\Delta p}{2m} \approx \frac{\sigma_p}{2m}$

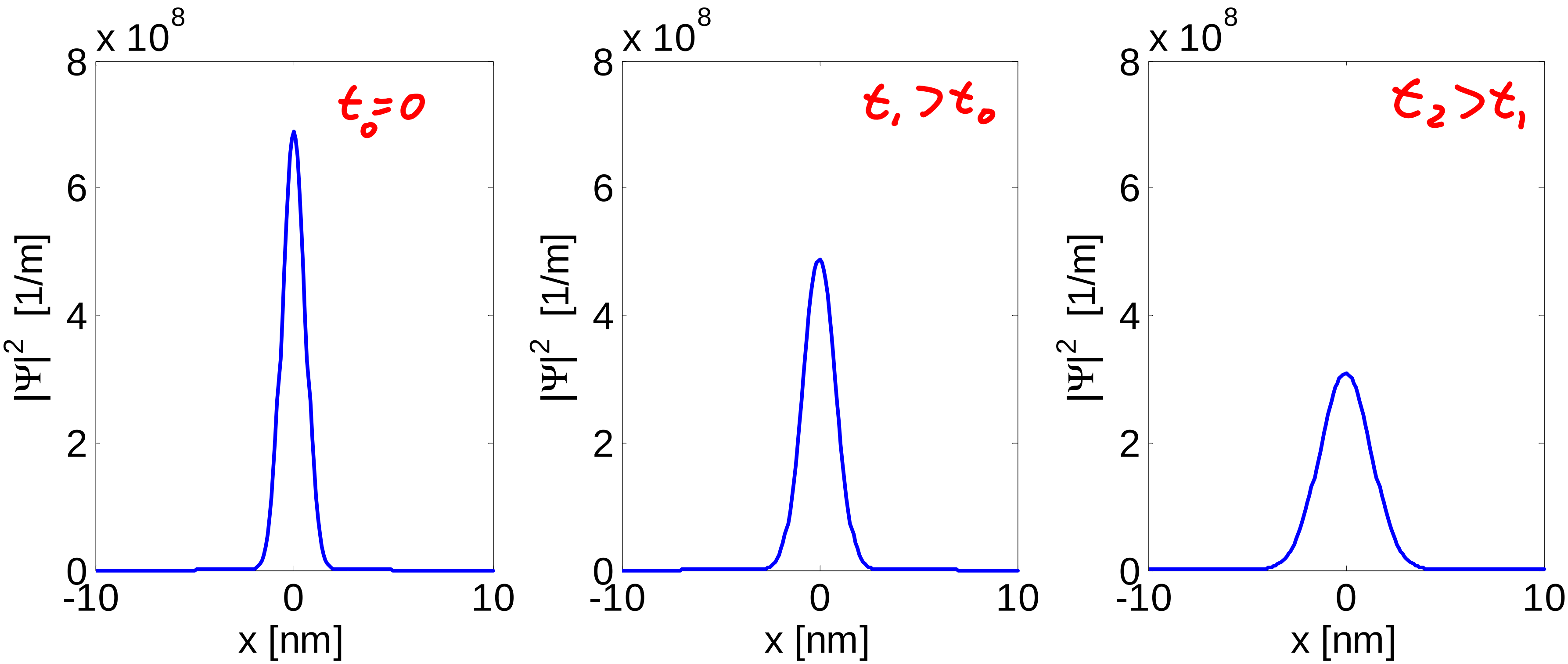
=) time scale for broadening $\approx \frac{\sigma_x|_{t=0}}{\Delta V_p} = \frac{\sigma_x|_{t=0}}{\sigma_p}2m \underset{\text{HUP}}{\approx} \frac{\hbar m}{\sigma_p^2}$

$\sigma_x\sigma_p \geq \hbar/2$

→ stationary Gaussian wave packet:



– stationary Gaussian wave packet:

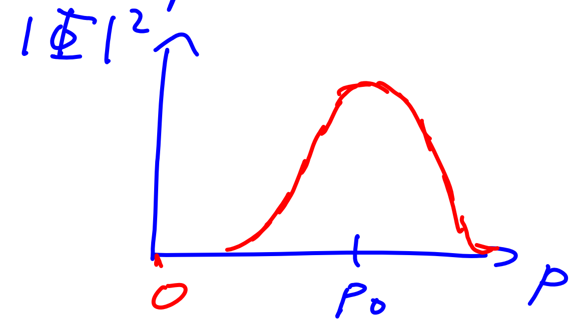


$\langle p \rangle = 0$ here $\Rightarrow$ wave packet does not move

$|\Phi(p,t)|^2$ is centered about $p_0 = 0$

Example II: Moving Gaussian wave packet

$\Rightarrow$ center $|\Phi(p,t)|^2$ about momentum $p_0 \neq 0$



$$\Rightarrow \Phi(p,t) = \frac{1}{(2\pi)^{1/4}\sqrt{\sigma_p}}\, e^{-\frac{(p-p_0)^2}{4\sigma_p^2}}\, e^{-\frac{ip^2}{2m}t/\hbar}$$

$\Rightarrow$ more math...

$$\Rightarrow \Psi(x,t) = \frac{1}{(2\pi)^{1/4}\sqrt{\hbar/2\sigma_p}}\ \frac{1}{\sqrt{1+i\frac{2\sigma_p^2}{m\hbar}t}}$$

$$\cdot \exp\left\{\frac{-\sigma_p^2/\hbar^2\left(x^2 - i\,\hbar/\sigma_p^2\, p_0 x + i\,\hbar/2\sigma_p^2 \cdot p_0^2\, t/m\right)}{1+i\frac{2\sigma_p^2}{m\hbar}t}\right\}$$

$$\Rightarrow |\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\left(\frac{\hbar}{2\sigma_p}\right)}\ \frac{1}{\sqrt{1+\left(\frac{2\sigma_p^2}{m\hbar}t\right)^2}}\ \exp\left\{-\frac{2\sigma_p}{\hbar^2}\ \frac{\left(x-\frac{p_0}{m}t\right)^2}{\left(1+\left(\frac{2\sigma_p^2}{m\hbar}t\right)^2\right)}\right\}$$

$\Rightarrow$ this is still a broadening Gaussian, as before, but:

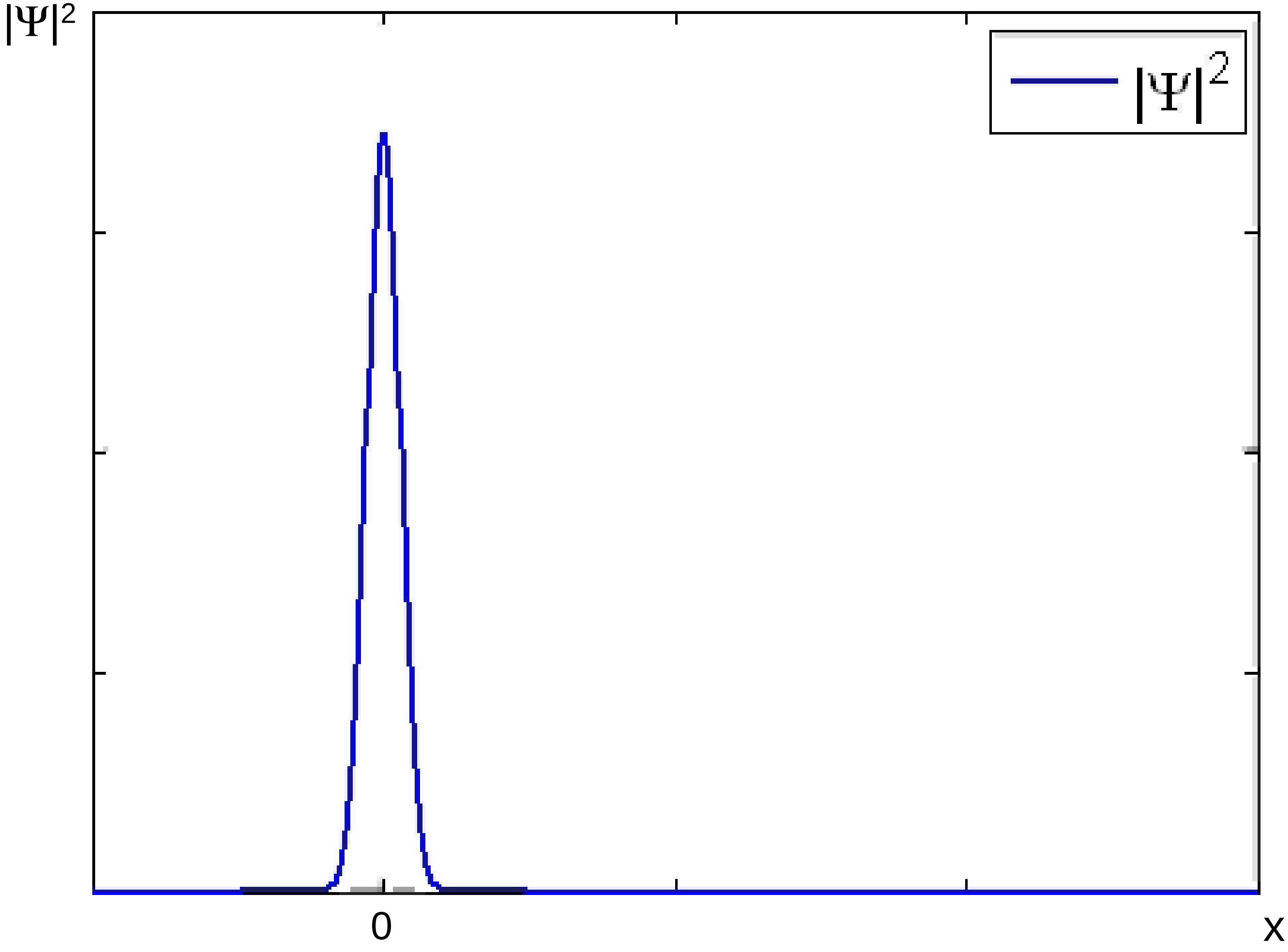
centered at $x - \frac{p_0}{m} t = 0 \Rightarrow$ moving!

$$x_{center} = \frac{p_0}{m} t = v_{classical} \cdot t$$

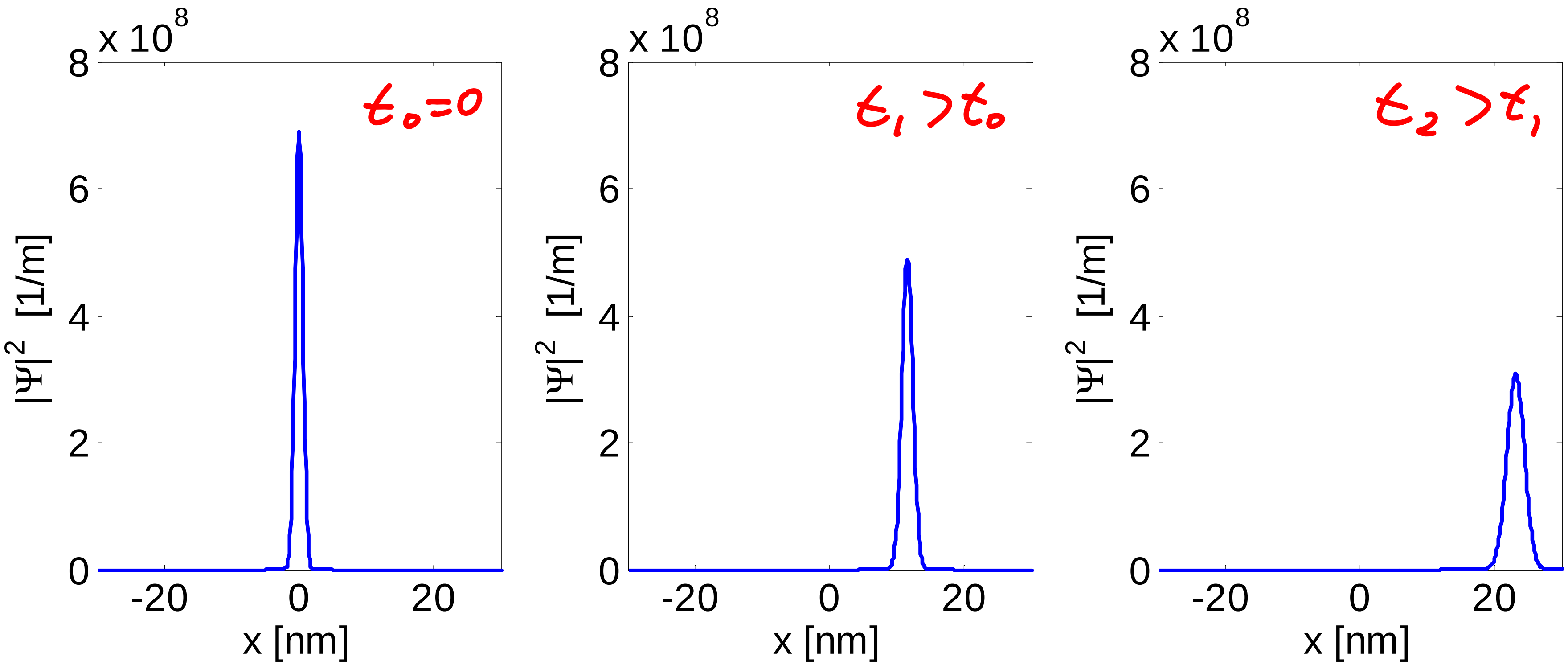
$$= v_{group}\, t$$

as expected!

→ moving Gaussian wave packet



→ moving Gaussian wave packet:



IV$_6$ Energy-Time Uncertainty Principle

position-momentum uncertainty
often written as: $\Delta x \, \Delta p \geq \hbar/2$

→ and paired with the energy-time uncertainty principle

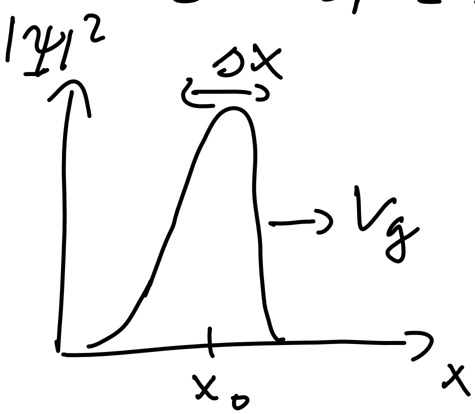
$$\Delta t \, \Delta E \geq \hbar/2$$

→ for free particle (CF&T)

$$\Delta x \cdot \Delta p = \Delta x \, \hbar \, \Delta k = v_g \Delta t \, \hbar \frac{\Delta \omega}{v_g} \geq \hbar/2$$

↑ $\Delta x = v_{group} \cdot \Delta t$

$$\Delta \omega \simeq \frac{d\omega(k)}{dk} \Delta k = v_g \Delta k$$



$$\Rightarrow \Delta t \, \hbar \Delta \omega = \boxed{\Delta t \, \Delta E \geq \hbar/2}$$

Energy-time uncertainty principle

Δt = time to pass point x_0

<u>But:</u> What is the full / generalized meaning of Δt?

$\rightarrow$ start with the time derivative of the expectation value of some observable $Q(x,p,t)$

$\Rightarrow$ measure of how fast the system is changing!

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle \Psi | \hat{Q} | \Psi\rangle$$

$$= \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q}\Psi\right\rangle + \left\langle \Psi \middle| \left(\frac{\partial \hat{Q}}{\partial t}\right)\Psi\right\rangle + \left\langle \Psi \middle| \hat{Q}\frac{\partial \Psi}{\partial t}\right\rangle$$

$\Rightarrow$ use Schrödinger's equation: $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V$$

$$\Rightarrow \frac{d}{dt}\langle Q\rangle = -\frac{1}{i\hbar}\langle \hat{H}\Psi | \hat{Q}\Psi\rangle + \left\langle \left(\frac{\partial \hat{Q}}{\partial t}\right)\right\rangle + \frac{1}{i\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi\rangle$$

$\rightarrow$ use that $\hat{H}$ is hermitian: $\langle \hat{H}\Psi | \hat{Q}\Psi \rangle = \langle \Psi | \hat{H}\hat{Q}\Psi \rangle$

[note: $\langle \Psi | \hat{H}\hat{Q}\Psi \rangle = \langle \hat{Q}\hat{H}\Psi | \Psi \rangle \neq \langle \hat{H}\hat{Q}\Psi | \Psi \rangle$

$$\Rightarrow \boxed{\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{Q}] \rangle + \left\langle \left(\frac{\partial \hat{Q}}{\partial t} \right) \right\rangle}$$