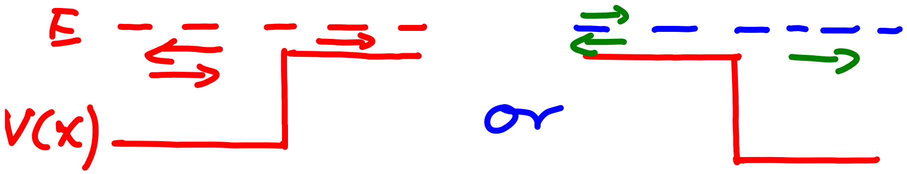
Lecture 35: 04/20/09

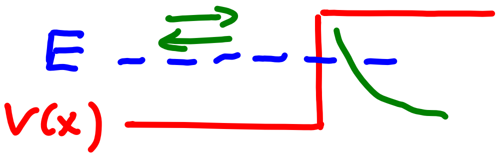
- Barrier penetration – tunneling
  - Examples
- Quantum Mechanics in 3-D

Recap

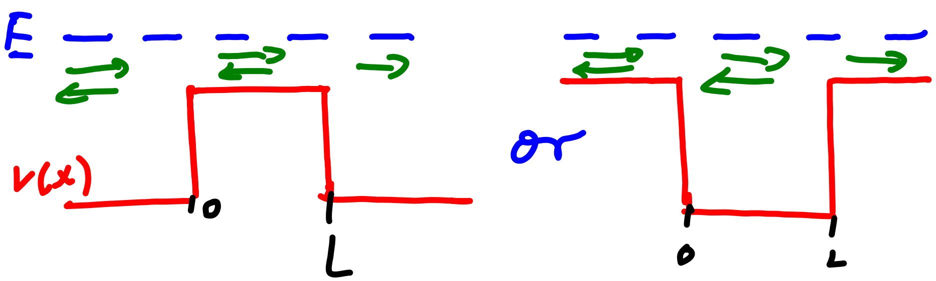
Particle scattering:



partial reflection at $V(x)$ jump

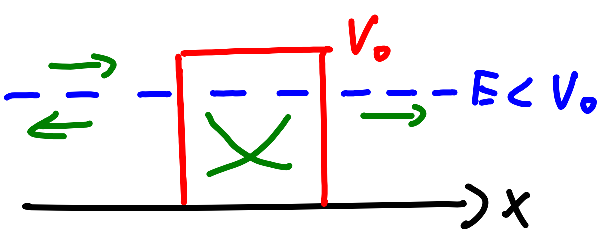


$R=1$; some penetration of ψ into classically excluded region



no reflection at certain resonant wavelengths/energies
$\left(L = n \cdot \lambda_2/2 \right)$

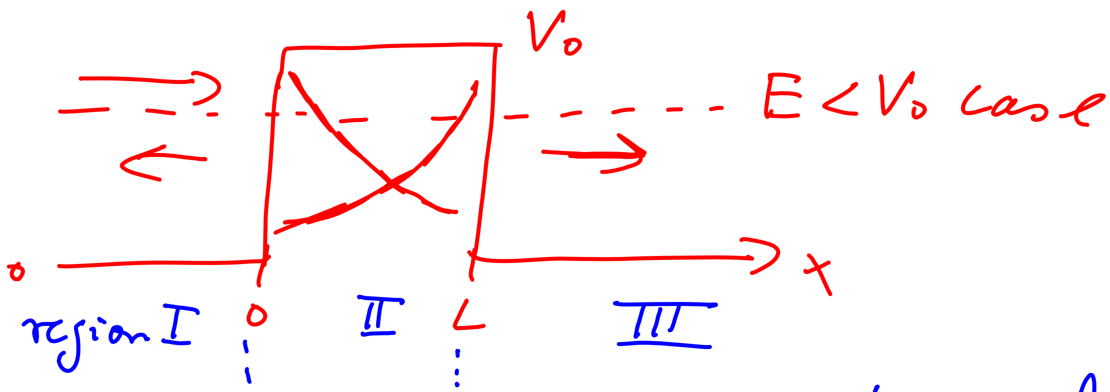
Tunneling:



$$T \approx \frac{16 k_1 \alpha^2}{(\alpha^2 + k_1^2)} e^{-2\alpha L} \qquad \text{for } \alpha L \gg 1$$

$\Rightarrow$ Quantum mechanical particle has probability >0 to "tunnel" through a potential barrier!

V_7 Barrier Penetration: Tunneling



$\rightarrow$ stationary states (solutions of time-indep. S.E)

$$\psi_w(x) = \begin{cases} A_0 e^{ik_1 x} + A e^{-ik_1 x} & \text{for } x \le 0 \\ B e^{-\alpha x} + C e^{+\alpha x} & \text{for } 0 < x \le L \\ D e^{ik_3 x} = D e^{ik_1 x} & \text{for } x > L \end{cases}$$

with $k_1 = k_3 = \dfrac{\sqrt{2mE}}{\hbar}$ and $\alpha = \dfrac{\sqrt{2m(V_0 - E)}}{\hbar}$ (real)

$\Rightarrow$ from boundary conditions at $x=0$ and $x=L$:

$$T = \left|\frac{D}{A_0}\right|^2 = \left| \frac{4ik_1\alpha}{\left[(i\alpha + k_1)^2 e^{+\alpha L} - (i\alpha - k_1)^2 e^{-\alpha L}\right] e^{ik_1 L}} \right|^2 \underline{\underline{> 0}}$$

$\uparrow$ $k_{in}/k_{out} = 1$

Note: – no resonant transmission if $E < V_0$

– for $T > 0$, need $J_p > 0$, also in region II

=> need B and $C \neq 0$ in $\Psi(x)$

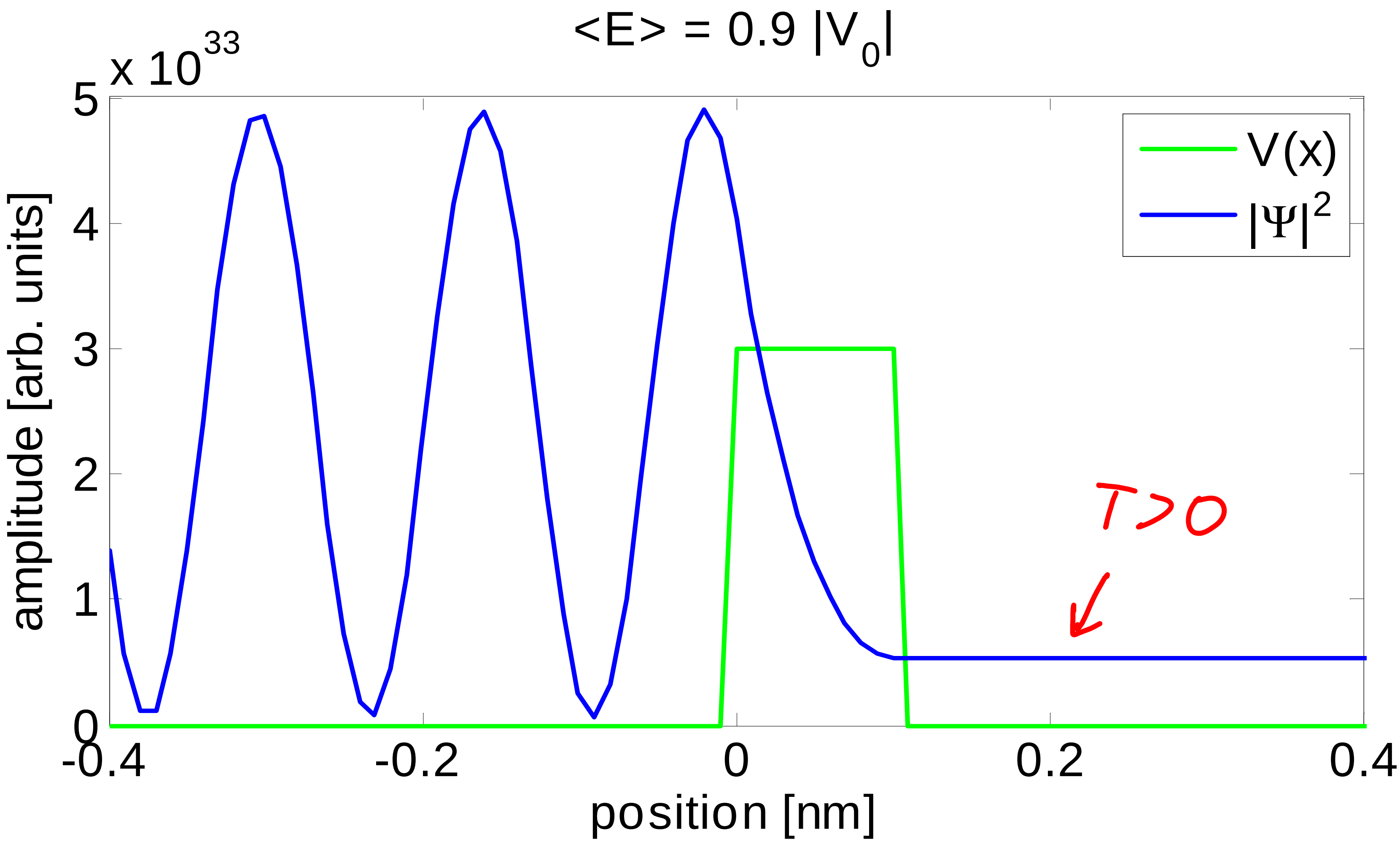
– tunneling: quantum mechanical wave can partially penetrate through a potential barrier that would block a classical particle!

– for a strong barrier: $\alpha L \gg 1$

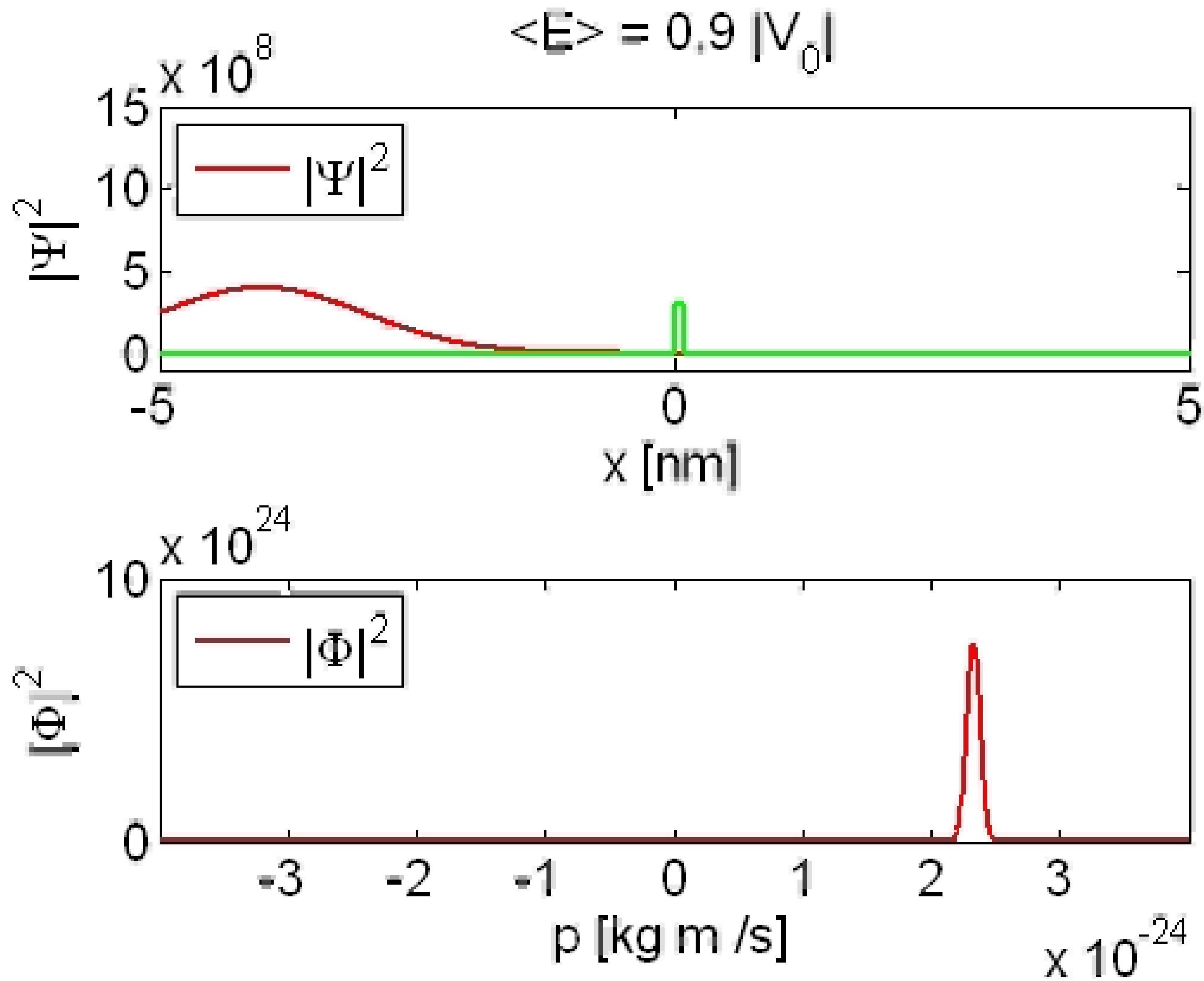
=> $e^{+\alpha L}$ - term dominates over $e^{-\alpha L}$ term:

$$\Rightarrow T = \left|\frac{D}{A_0}\right|^2 \approx \underbrace{\frac{16 k_1^2 \alpha^2}{(\alpha^2 + k_1^2)}}_{\text{weak function of energy, typical of order } 1} \underbrace{e^{-2\alpha L}}_{e^{-2\frac{\sqrt{2m(V_0 - E)}}{\hbar}L}} \quad \text{for } \alpha L \gg 1$$

exponential decay of tunneling probability with barrier thickness L

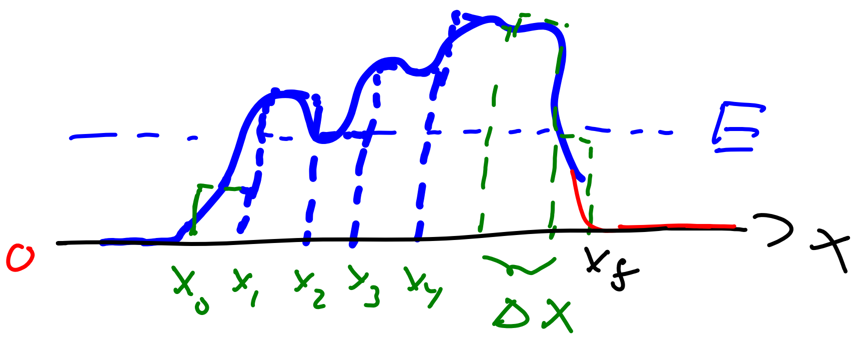


<E> = 0.9 |V0|
x 10^33
5
4
3
2
1
0
-0.4
-0.2
0
0.2
0.4
amplitude [arb. units]
position [nm]
V(x)
|Ψ|^2
T>0



<E> = 0,9 |V0|
x 10^8
15
10
5
0
|Ψ|^2
-5
0
5
x [nm]
x 10^24
10
0
|Φ|^2
-3
-2
-1
0
1
2
3
p [kg m /s]
x 10^-24

→ for arbitrary $V(x)$:



approximate the barrier as a series of square-like barriers with exponential decay accross each one

$$T_{\Delta x} \approx e^{-2\frac{\sqrt{2m(V(x)-E)}}{\hbar}\cdot \Delta x}$$

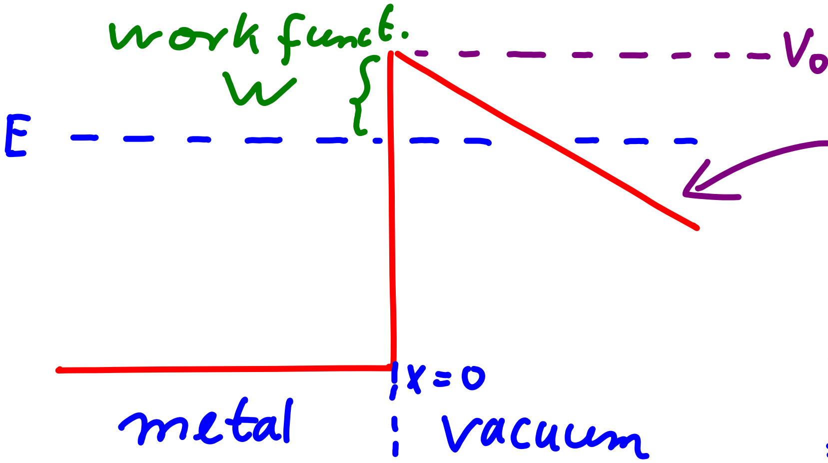
$$\Rightarrow T_{total} \approx 1 \cdot e^{-2\frac{\sqrt{2m(V(x_1)-E}}{\hbar}\Delta x} \cdot e^{-2\frac{\sqrt{2m(V(x_2)-E}}{\hbar}\Delta x} \cdots$$

$$= e^{-2\frac{\sqrt{2m}}{\hbar}\left\{\sqrt{V(x_1)-E}\,\Delta x + \sqrt{V(x_2)-E}\,\Delta x + \cdots\right\}}$$

$$\underbrace{\approx e^{-2\frac{\sqrt{2m}}{\hbar}\int_{x_0}^{x_f}\sqrt{V(x)-E}\,dx}}_{\text{good formula for making order of magnitude estimats for } T}$$

Examples of tunneling

Example 1: Field emission of electrons



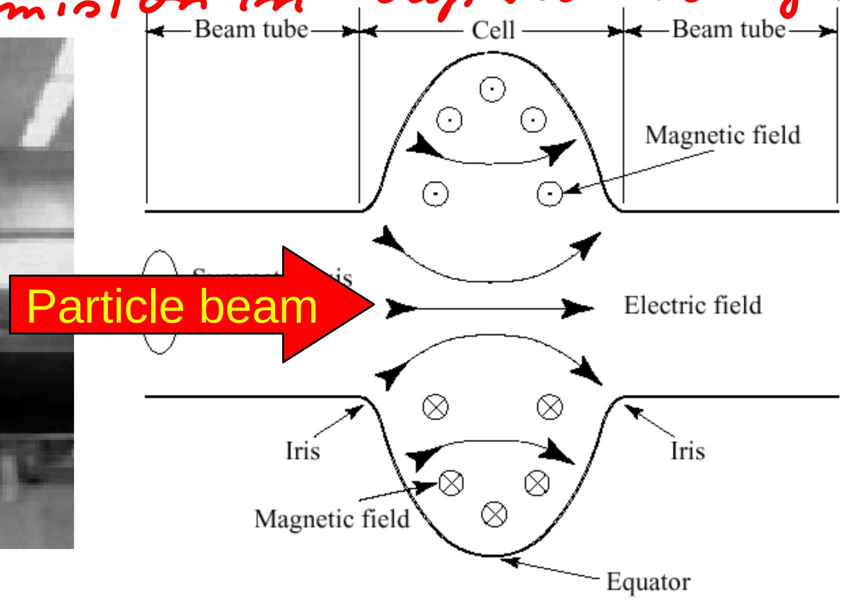
high external electric field $\mathcal{E}$

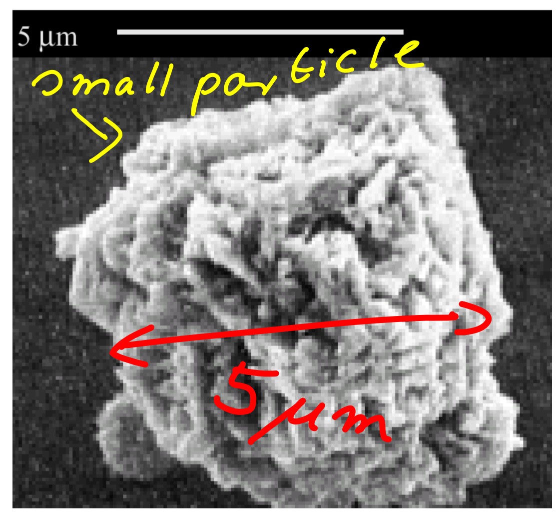
$\Rightarrow V(x) = V_0 - e\mathcal{E}x \quad (x>0)$

$$\Rightarrow T \approx \exp\left\{-\frac{4}{3}\frac{\sqrt{2m}}{\hbar}\frac{W^{3/2}}{e\mathcal{E}}\right\}$$

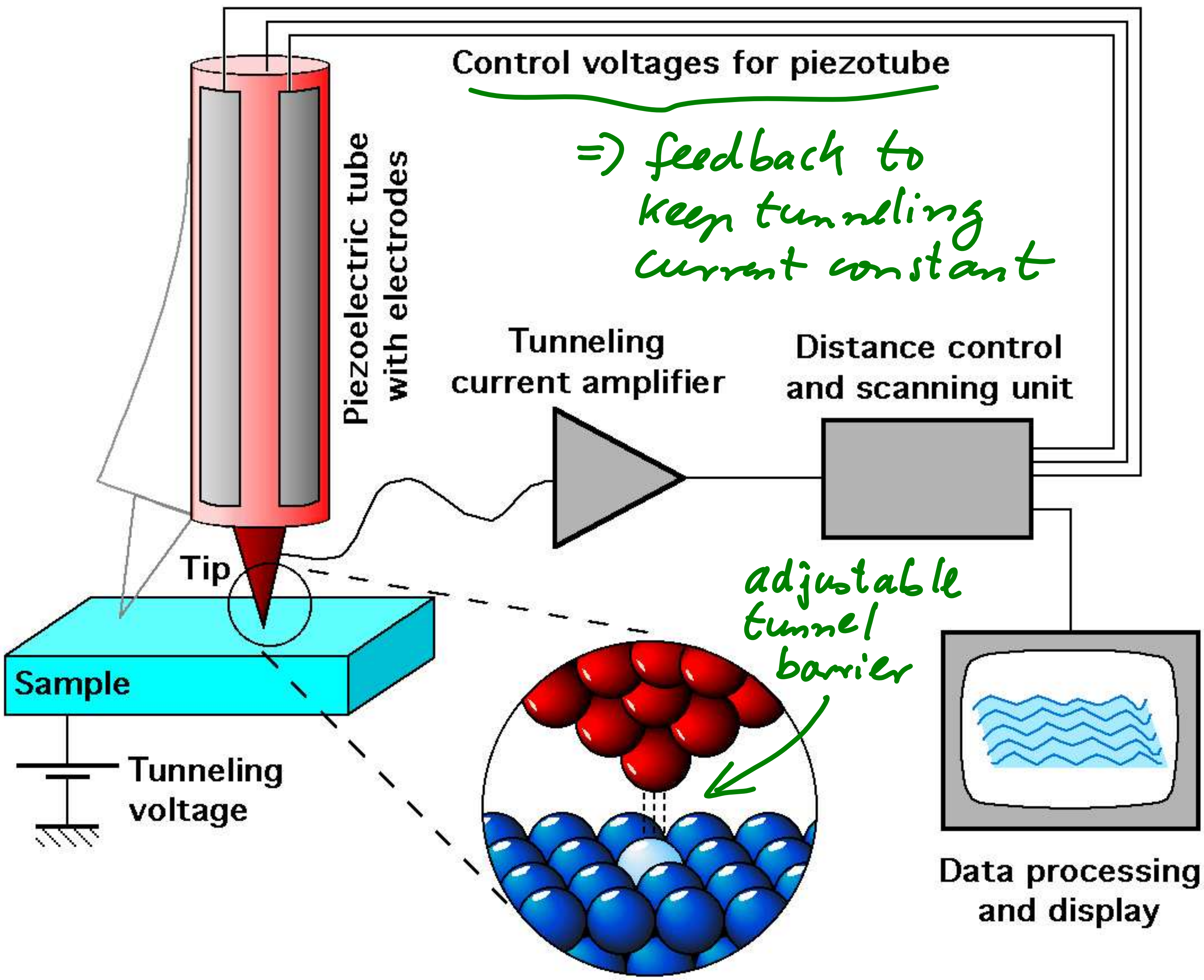
$\Rightarrow$ need very high fields $\mathcal{E} \gtrsim 10^9\ V/m$

Example: Field emission in superconducting RF accelerator cavities:





Example 2: Scanning Tunneling Microscope (STM)



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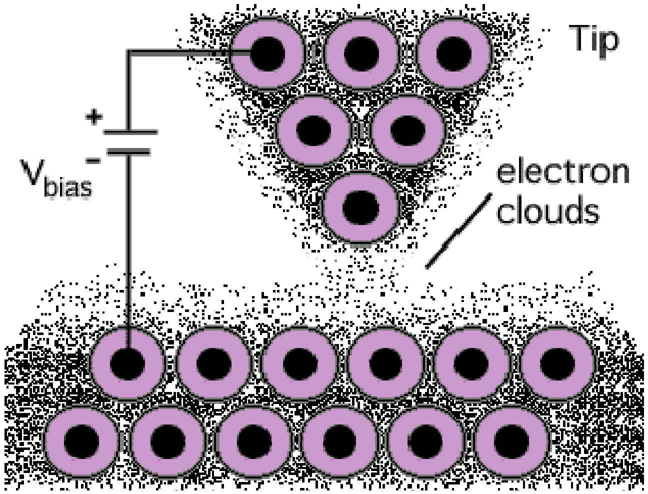
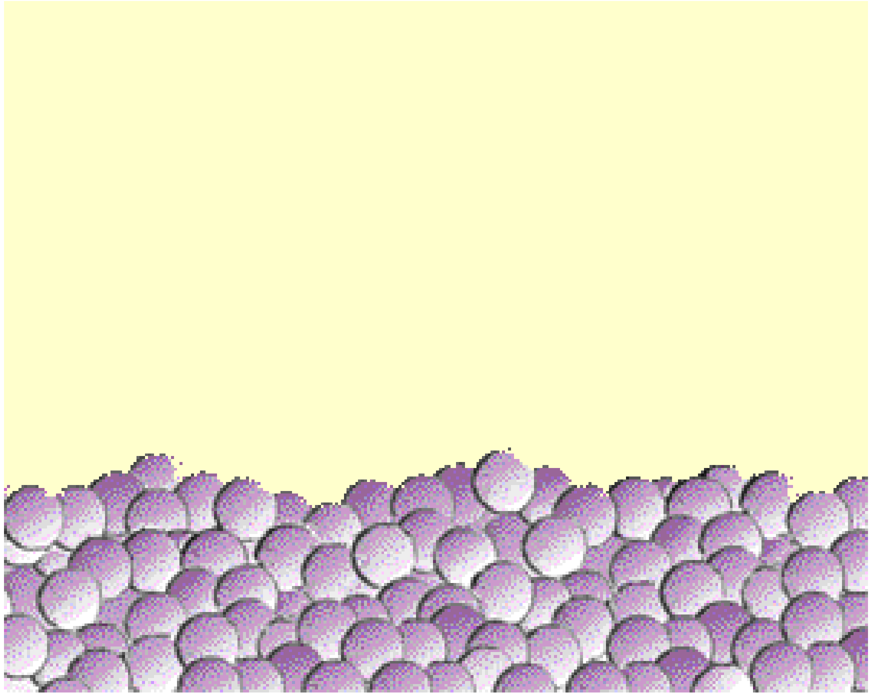
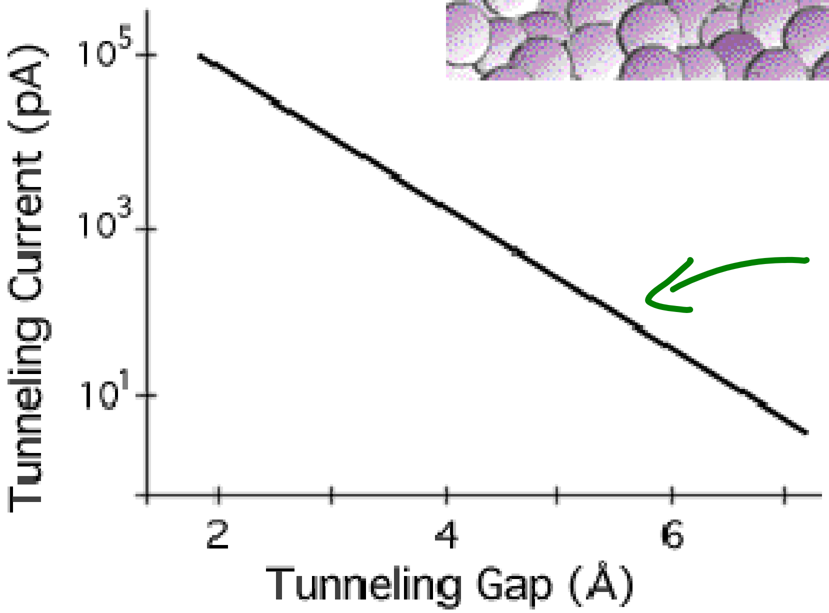
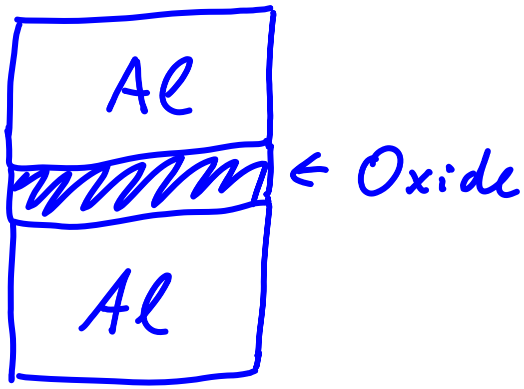






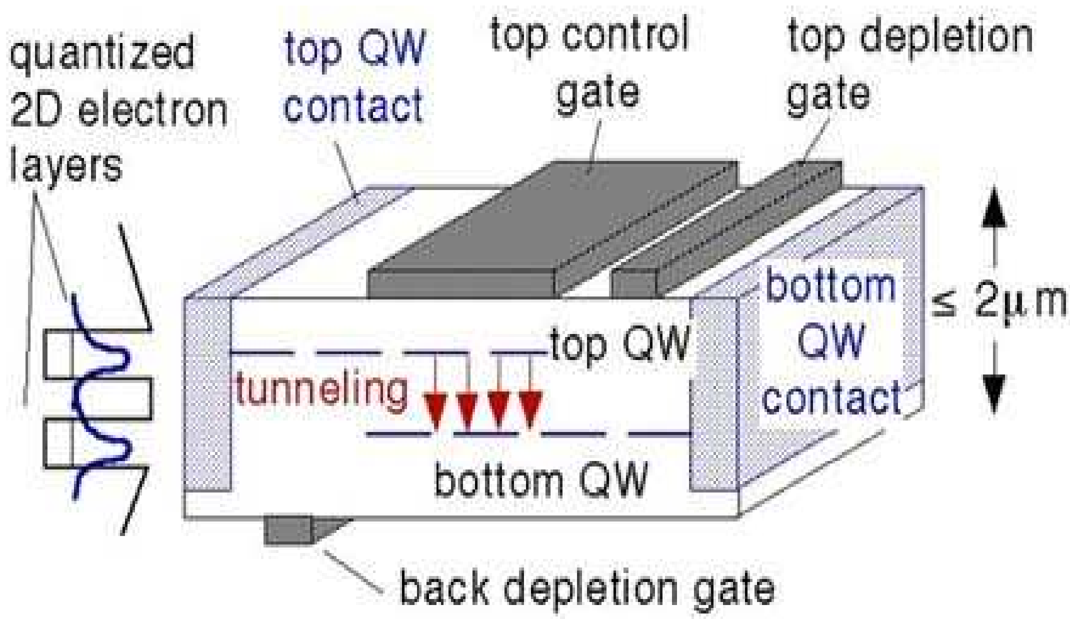
exponential dependence of current on distance

Example 3: Tunneling Junctions



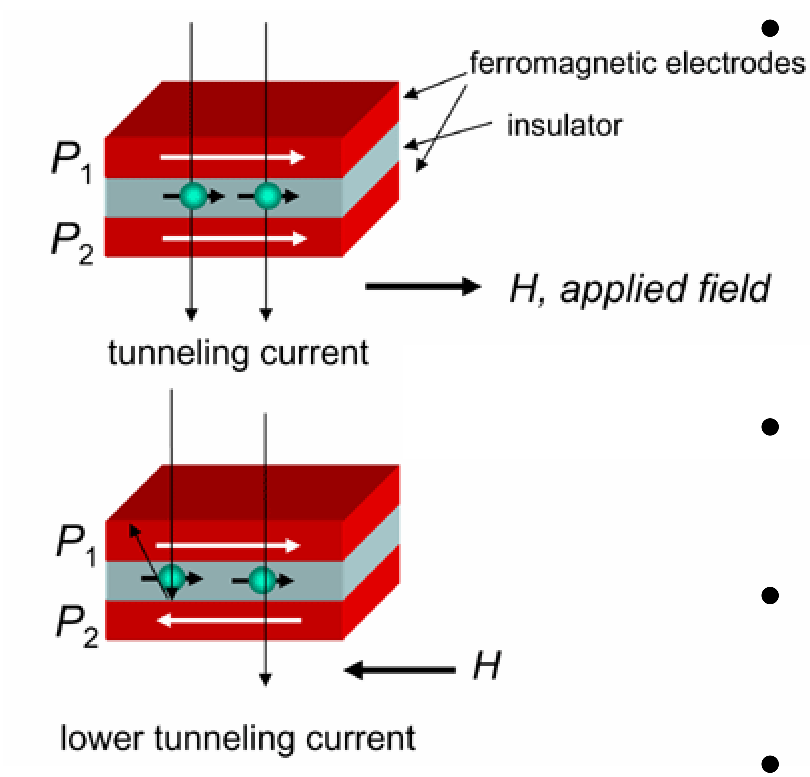
- Electrons can tunnel through oxide barrier
- Resistance varies exponentially with oxide thickness
- ⇒ multi-junction devices...

Example 4: Tunneling Transistors



- *Upper quantum well (labelled "top QW") and the lower quantum well ("bottom QW"), are made of gallium arsenide (thicknesses of <2 μm)*
- *Adjusting the voltage allows the electrons in the top QW to "tunnel through" an ordinarily insurmountable barrier (made of aluminum gallium arsenide to the bottom QW.*
- *Tunneling occurs when the top QW and bottom QW accept electrons with the same energy states.*

Example 5: Tunnel Magnetoresistance (TMR)



- **A TMR device consists of two ferromagnetic layers (red) separated by an insulating spacer (less than about 2 nm, grey).**
- **Tunnel current can flow between the ferromagnets.**
- **Resistance depends on the relative magnetic orientation**
- **Sensitive magnetic field sensor!**
- **Used in hard drives to read information from discs**

VI Quantum Mechanics in 3-D:

VI$_1$ Schrödinger's Equation in 3-D

→ extend time-dep. Schrödinger Equation to 3-D

$$i\hbar \frac{\partial \Psi(x,y,z,t)}{\partial t} = \hat{H}_{3D}\,\Psi(x,y,z,t)$$

→ recall: in 1-D

classical energy

$$E = \frac{1}{2} m v_x^2 + V(x)$$

$$= \frac{p_x^2}{2m} + V(x)$$

in QM: Hamiltonian operator

$$\longrightarrow \hat{H} = \frac{\hat{p}_x^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

→ extend this to 3-D:

- classical energy

$$E = \frac{1}{2}mv^2 + V = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x,y,z)$$

- use: $p_x \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}$ $\quad p_y \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial y}$ $\quad p_z \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial z}$

⇒ Hamiltonian operator:

$$\hat{H}_{3D} = -\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right\} + V(x,y,z)$$

$$= -\frac{\hbar^2}{2m}\nabla^2 + V(x,y,z)$$

where: $\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} \equiv \left.\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right\}$ "Laplacien" in cartesian coordinats

$\Rightarrow$ 3-D time-dep. Schrödinger Equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(x,y,z,(t))\,\Psi(x,y,z,t)$$

$\rightarrow$ units:

1-D: $\int_{\Delta x} |\Psi|^2 dx$ = probability $\Rightarrow [\Psi] = \frac{1}{\sqrt{m}}$

3-D: $\int_{\Delta V} |\Psi|^2 dx\,dy\,dz$ = probability $\Rightarrow [\Psi] = \frac{1}{(m)^{3/2}}$

$\rightarrow$ Normalization condition for 3-D

$$\int_{\text{all space}} |\Psi|^2 dx\,dy\,dz = 1$$

$\rightarrow$ similar for inner products:

$$\langle f | g \rangle = \int_{\text{all space}} f^* g \; dx\,dy\,dz$$

→ if $V(\vec{r})$ is time-independent, there will be a complete, orthonormal set of stationary states (i.e. states with definite energy!)

$$\Psi_n(x,y,z,t) = \underbrace{\psi_n(x,y,z)}_{\text{spatial wave function}} \underbrace{e^{-i\frac{E_n}{\hbar}\cdot t}}_{\text{time dependence}}$$

where $\psi_n(x,y,z)$ is a solution of the time-indep. Schrödinger Equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi_n + V(x,z,y)\psi_n = E_n\psi_n(x,y,z)$$

or short: eigenvalue equation): $\hat{H}_{3D}\psi_n = E_n\psi_n$

→ general solution of the time-dep. Schrödinger equation:

$$\Psi(x,y,z,t) = \sum_n c_n \psi_n(x,y,z)\, e^{-i\frac{E_n}{\hbar}t}$$

where coefficients c_n are determined by the initial wave function $\Psi(x,y,z,t=0)$

$$c_n = \langle \psi_n | \Psi(x,y,z,t=0) \rangle$$