Lecture 37: 04/24/09

- Orbital angular momentum of particles in QM

Recap

VI$_2$ Separable Potentials

$$V = V_1(x) + V_2(y) + V_3(z)$$

$\Rightarrow$ solution: $\psi = \underbrace{\psi_1(x)}_{\text{solution of 1-D S.E}} \psi_2(y) \psi_3(z)$ $\qquad E = E_1 + E_2 + E_3$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_1(x)}{\partial x^2} + V_1(x)\psi_1(x) = E_1 \psi_1(x)$$ 1-D time-indep. S.E!

$\Rightarrow$ similar for $\psi_2(y)$ and $\psi_3(z)$

VI$_3$ Central Potentials

$V = V(r)$

Recap

try: $\psi(r, \theta, \phi) = R(r) F(\theta, \phi) = \frac{u(r)}{r} F(\theta, \phi)$

① Differential equ. for angular function $F(\theta, \phi)$

$$\underbrace{-\hbar^2 \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right\}}_{\text{operator } \hat{L^2} \text{ corresponds to } L^2 \text{ in classical physics}} F(\theta, \phi) = \hbar^2 \beta F = \underset{\text{eigenvalue}}{L^2} F(\theta, \phi)$$

② Differential equ. for radial wave function $u(r)$

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left\{ \frac{L^2}{2mr^2} + V(r) \right\} u(r) = E\, u(r)$$

(i) Energy of the particle is determined by ②

(ii) Equation ① does not depend on $V(r)$

(iii) functions $F(\theta, \phi)$ are eigenfunctions of $\hat{L^2}$, i.e. they have definite values of $L^2 = \hbar^2 \beta$

in 2

$$V_{eff} = \frac{\hbar^2 \beta}{2 m r^2} + V(r) \quad : \text{effective potential}$$

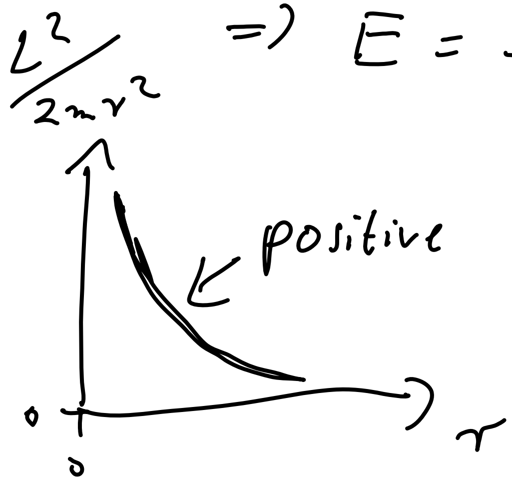
$\Rightarrow$ compare to classical physics:

$$E = \frac{\vec{p}^{\,2}}{2m} + V = \underbrace{\frac{1}{2}\frac{p_r^2}{m}}_{\text{radial component}} + \underbrace{\frac{1}{2}\frac{(p_t)^2}{m}}_{\text{transverse component}} + V$$

angular momentum: $\vec{L} = \vec{r} \times \vec{p}$

$$\Rightarrow L^2 = r^2 p_t^2 \Rightarrow p_t^2 = \frac{L^2}{r^2}$$

$$\Rightarrow E = \frac{1}{2}\frac{p_r^2}{m} + \frac{L^2}{2 m r^2} + V$$



$\Rightarrow$ conclusion: $\frac{\hbar^2 \beta}{2 m r^2}$ - term should be related to extra kinetic energy due to transverse "motion" with constant angular momentum L

$\Rightarrow$ $\hbar^2\beta$ should be related to L^2!

$\Rightarrow$ assume $\underline{L^2 = \hbar^2\beta} = (\text{angular momentum})^2$

$\Rightarrow$ with ①:

eigenfunction

$$\underbrace{-\hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}}_{\text{operator } \hat{L}^2 \text{ corresponding to } L^2 \text{ in classical physics!}} F(\theta,\phi) = \hbar^2\beta F = L^2 F(\theta,\phi)$$

eigenvalue

$\Rightarrow$ eigenvalue equation: $\boxed{\hat{L}^2 F(\theta,\phi) = L^2 F(\theta,\phi)}$

$\Rightarrow$ functions $F(\theta,\phi)$ are eigenfunctions of operator $\hat{L}^2$, i.e. they have definite values of $L^2 = \hbar^2\beta$!

$\Rightarrow$ $\sigma_{L^2} = 0$

$\Rightarrow$ can use this to label solutions F!

(iiii) Subset of solutions:

sherically symmetric

$\Rightarrow$ wave function $F(\theta, \phi) = \text{const}$

$\Rightarrow$ $L^2 = 0$ from 1

$\Rightarrow$ can ignore the angular momentum term in 2 $\Rightarrow$ see F&T, chapter 5

- **Central Potentials:** $V = V(r)$

$$\psi(r,\theta,\phi) = R(r)\,F(\theta,\phi) = \frac{u(r)}{r}\,F(\theta,\phi)$$

① Differential equ. for angular function $F(\theta,\phi)$

$$\underbrace{-\hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}}_{\text{operator } \hat{L}^2 \text{ corresponds to } L^2 \text{ in classical physics}}F(\theta,\phi) = \hbar^2\beta F = L^2\,F(\theta,\phi)$$

eigenvalue

② Differential equ. for radial wave function $u(r)$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left\{\frac{L^2}{2mr^2} + V(r)\right\}u(r) = E\,u(r)$$

VI_4 Orbital Angular Momentum of Particles in QM

→ classically:

$$\vec{L} = \vec{r} \times \vec{p} \leftarrow \text{linear momentum}$$

position vector

$$= \begin{cases} \vec{i}\,(y p_z - z p_y) \\ \vec{j}\,(z p_x - x p_z) \\ \vec{k}\,(x p_y - y p_x) \end{cases}$$

unit vector along x, y, z

in QM: → $\frac{\hbar}{i}\frac{\partial}{\partial x}$

→ in QM: operator (for position space!)

$$\hat{L} = \hat{r} \times \hat{p} = \begin{cases} \vec{i}\,\frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \leftarrow \text{component along x-direction} = \hat{L}_x \\ \vec{j}\,\frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \leftarrow \text{component along y- "} = \hat{L}_y \\ \vec{k}\,\frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \leftarrow \text{" " z- "} = \hat{L}_z \end{cases}$$

→ transform to spherical coordinats (r, θ, ϕ)

example: $\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$

Proof: $z = r\cos\theta$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$$\frac{\hbar}{i}\frac{\partial}{\partial \phi} = \frac{\hbar}{i}\left[\frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z}\right]$$

↑ chain rule of implicit differentiation

$$= \frac{\hbar}{i}\left[(-y)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + 0\,\frac{\partial}{\partial z}\right]$$

$$= \frac{\hbar}{i}\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right] = \hat{L}_z \quad ✓$$

$\Rightarrow$ similar (see F&T, page 477-479)

$$\hat{L}_x = \frac{\hbar}{i}\left(-\sin\phi \frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_y = \frac{\hbar}{i}\left(\cos\phi \frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$\Rightarrow$ calculate operator $\hat{L^2}$ corresponding to L^2 in classical physics: $L^2 = L_x^2 + L_y^2 + L_z^2$

$\hat{L^2} = \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y + \hat{L}_z\hat{L}_z = \ldots\ldots$ lots of algebra...

$$= -\hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}$$

<u>as before!</u>

→ eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$:

- $\hat{L}^2$: from before: $F(\theta,\phi)$ is an eigenfunction of $\hat{L}^2$

$$\hat{L}^2 F(\theta,\phi) = \hbar^2 \Lambda F(\theta,\phi) = L^2 F(\theta,\phi)$$

$\Rightarrow$ angular parts $F(\theta,\phi)$ of solution of the time indep. S.E. for a central potential $V(r)$ are eigenfunctions of $\hat{L}^2$, i.e. have definite values of L^2 and energy E

- $\hat{L}_z$: $\hat{L}_z \psi = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\psi = L_z \psi$ for eigenfunctions of $\hat{L}_z$

eigenvalue!

$$\Rightarrow \frac{\partial \psi}{\partial \phi} = i\frac{L_z}{\hbar}\psi \Rightarrow \psi \propto e^{i\frac{L_z}{\hbar}\phi} \text{ : eigenfunction}$$

$\Rightarrow$ states of definite L_z

now: when ϕ advances by 2π $\Rightarrow$ return to same point in space!

$\Rightarrow$ require: $\psi(\phi + 2\pi) \overset{!}{=} \psi(\phi)$

$$\Rightarrow e^{i \frac{L_z}{\hbar}(\phi + 2\pi)} = e^{i \frac{L_z}{\hbar}\phi}$$

$$\Rightarrow e^{i L_z/\hbar \cdot 2\pi} \overset{!}{=} 1 \quad \Rightarrow \frac{L_z}{\hbar} 2\pi = 2\pi \cdot \text{integer}$$

$\Rightarrow L_z = \hbar \cdot \text{integer} = \hbar m \qquad m = 0, \pm 1, \pm 2 \ldots$

$\Rightarrow$ quantized! $\Rightarrow$ eigenfunctions $\psi \propto \underline{e^{im\phi}}$

$\Rightarrow$ if one measures the z-component of the angular momentum of a particle

$\Rightarrow$ will only get values $L_z = m\hbar \quad m = 0, \pm 1, \pm 2 \ldots$

⇒ by symmetry: same should apply to $\hat{L}_x$ and $\hat{L}_y$

⇒ L_x and L_y are quantized too!

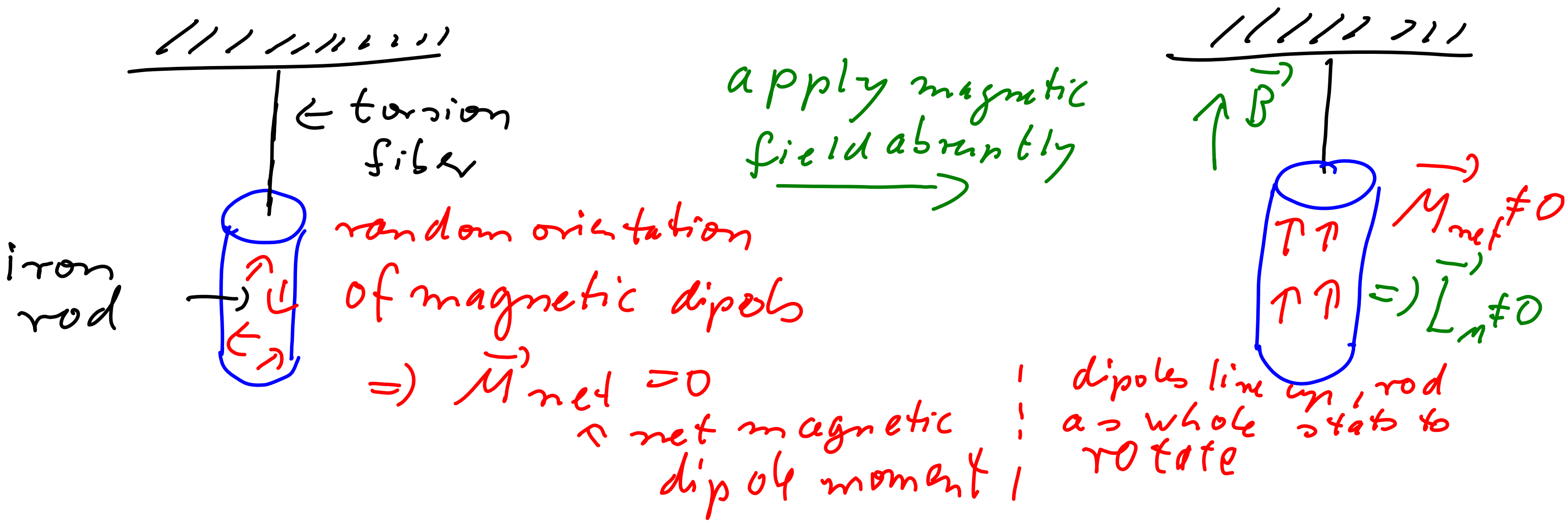
⇒ only possible results of measurements on any particle: $L_x = m_x \hbar \quad m_x = 0, \pm 1, \ldots$

$L_y = m_y \hbar \quad m_y = 0, \pm 1, \ldots$

→ Experimental proof:

① Einstein – de Haas effect:

⇒ shows that magnetic moments are related to angular momentum



Explanation:

$$\underset{\substack{\text{net}\\ \text{magnetic dipole}\\ \text{moment}}}{\vec{\mu}_{net}} = \underset{\substack{\Uparrow\\ \text{constant}\\ (<0 \text{ here})}}{C} \; \underset{\substack{\text{angular momentum}\\ \text{of atoms}}}{\vec{L}_{\mu}}$$

$\Rightarrow$ angular momentum is conserved

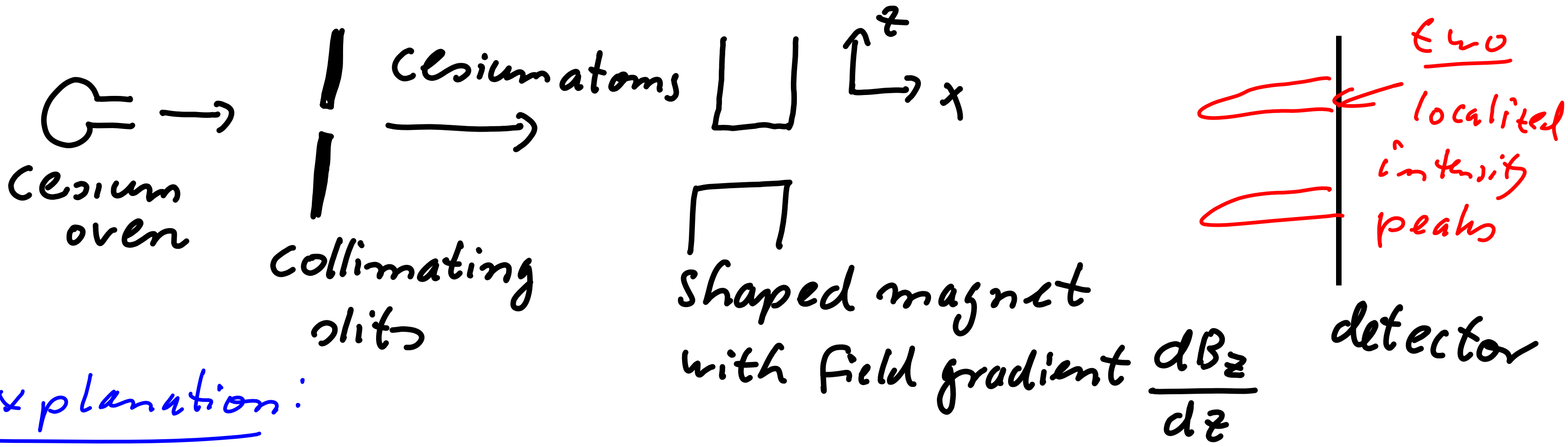
before: $L_\mu = 0$ $\quad L_{rod} = 0 \Rightarrow L_\mu + L_{rod} = 0$

after: $L_\mu \neq 0 \Rightarrow L_{rod} = -L_\mu = -\frac{\mu}{|C|} > 0$

$\Rightarrow$ starts to rotate!

② Stern-Gerlach Experiment (1922)

→ shows that magnetic moments of atoms are quantized ⇒ angular momentum is too



Explanation:

non-constant B-field ⇒ magnetic dipoles of the atoms feel a linear force

$$F_z = \mu_z \frac{dB_z}{dz}$$

↑ dipole moment along z-direction

$\Rightarrow$ intensity distribution on the detector gives distribution of μ_z of the atoms

$\Rightarrow$ μ_z for the cesium atoms is quantized!

$\Rightarrow$ angular momentum is quantized!