Lecture 10

3. Linear transverse beam optics

3.7 Phase space ellipse

3.8 Emittance and beam envelope

Twiss Parameters

particle trajectory: $u = x$ $u = z$

$$u(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

where J and ϕ_0 given by starting position u_0, u_0'

Optical functions: defined by beam optics + initial $\alpha(s_0)$, $\beta(s_0)$, $\gamma(s_0)$ [from initial beam distribution or periodicity conditions in a circular accelerator]

$$\beta'(s) = -2\alpha(s)$$

$$\alpha'(s) = K(s)\beta(s) - \gamma(s)$$

$$\gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

$$\psi(s) = \int_0^s \frac{1}{\beta(\tilde{s})}\,d\tilde{s}$$

$[\beta] = m$, typically several m to several 10 m

$[\alpha] = 1$

$[\gamma] = \frac{1}{m}$

$\Rightarrow$ since $\psi' = \frac{1}{\beta} = \frac{d\psi}{ds} \Rightarrow d\psi = \frac{1}{\beta} ds = \frac{2\pi}{\lambda} ds$

$\Rightarrow$ local wavelength of quasi-harmonic motion : $\lambda(s) = 2\pi \beta(s)$

$\Rightarrow$ of order of several 10 meters

3.7 Phase space ellipse

particle trajectory: $u(s) = \sqrt{2J\beta(s)} \sin(\psi(s) + \phi_0)$

$\Rightarrow$ consider family of trajectoris with same amplitude $\sqrt{2J}$ but different phase ϕ_0

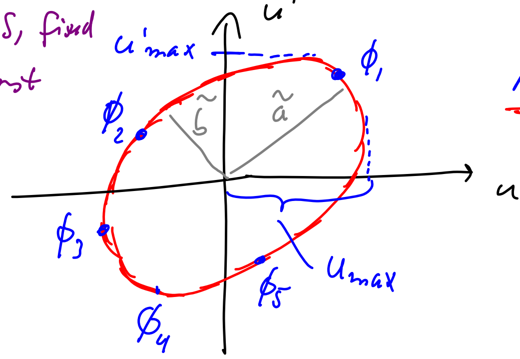
$\Rightarrow$ at fixed value of s, plot vector $\mathcal{U}(\phi_0) = (u, u')$

with

parametric representation of ellipse in (u, u') plane

$$\begin{cases} u = \sqrt{2J\beta} \sin(\psi + \phi) \\ u' = \frac{\sqrt{2J}}{\sqrt{\beta}} \left[\cos(\psi + \phi) - \alpha \sin(\psi + \phi)\right] \end{cases}$$

at $s=s_1$ find
$J = \text{const}$



area of ellipse:

$$A = \int_0^{2\pi}\int_0^{J} dJ\, d\phi$$

$$= \iint du\, du'$$

$$= \pi \tilde{a}\tilde{b}$$

$$= \pi\, 2J$$

$[A] = \pi$ m rad

$$u_{max} = \sqrt{2J\beta} \quad \text{at} \quad u' = -\alpha\sqrt{\frac{2J}{\beta}}$$

$$u'_{max} = \sqrt{\frac{2J}{\beta}}\sqrt{1+\alpha^2} = \sqrt{2J\gamma} \quad \text{at} \quad u = -\alpha\sqrt{\frac{2J}{\gamma}}$$

$$\Rightarrow \text{from} \quad u(s) = \sqrt{2J}\sqrt{\beta}\,\sin(\psi+\phi) \qquad (a)$$

$$u'(s) = \frac{\sqrt{2J}}{\sqrt{\beta}}\left[\cos(\psi+\phi) - \alpha\sin(\psi+\phi)\right] \qquad (b)$$

– (a) gives: $\sin(\psi+\phi) = \dfrac{u}{\sqrt{2J\beta}}$

– insert into (b)

$$\cos(\psi+\phi) = \frac{\sqrt{\beta}\,u'}{\sqrt{2J}} + \frac{\alpha\, u(s)}{\sqrt{2J}\sqrt{\beta}}$$

– using $\sin^2\theta + \cos^2\theta = 1$ gives

$$\frac{u^2}{2J\beta} + \left(\frac{\alpha\, u}{\sqrt{2J}\sqrt{\beta}} + \frac{\sqrt{\beta}}{\sqrt{2J}}u'\right)^2 = 1$$

$$\Rightarrow \frac{u^2}{\beta} + \left(\frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}\,u'\right)^2 = 2J$$

$\Rightarrow$ with $\gamma(s) = \dfrac{1+\alpha^2(s)}{\beta(s)}$

$\Rightarrow f(u,u',s) = \gamma(s)\,u^2(s) + 2\alpha(s)\,u(s)\,u'(s) + \beta(s)\,u'(s)^2 = 2J$

= const through motion along beam line!

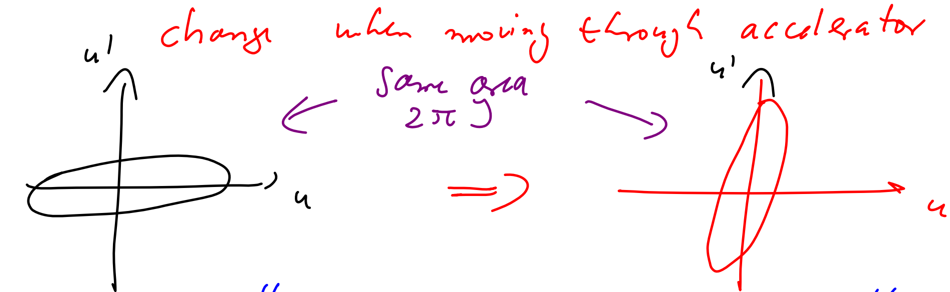
= Courant-Snyder invariant

= area of ellipse / π

$\Rightarrow \dfrac{df}{ds} = 0$

$\Rightarrow$ area of phase space ellipse is invariant!

$\Rightarrow$ shape and orientation of ellipse change when moving through accelerator



at position with large β

$u \propto \sqrt{\beta}$ large

$u' \propto \frac{1}{\sqrt{\beta}}$ small

at position with small β

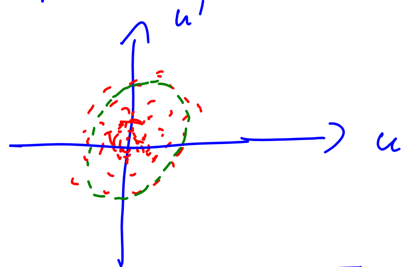
$u \propto \sqrt{\beta}$ small

$u' \propto \frac{1}{\sqrt{\beta}}$ large

3.8 Emittance and beam envelope

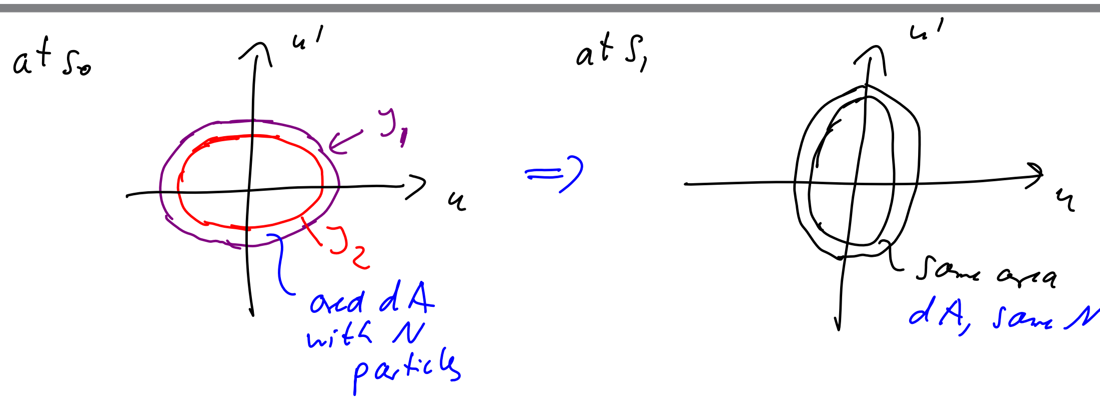
now: consider particle beam with many particles and a distribution of initial parameter J, ϕ_0

$\Rightarrow$ particles fill the (u, u') phase space at a certain point s:



$\Rightarrow$ trajectories: $u(s) = \sqrt{2J\beta}\,\sin(\psi + \phi_0)$

$\Rightarrow$ areas of phase space ellipses are constant!



$\Rightarrow$ phase space density $n = \dfrac{dN}{dA} = \text{const}$!

Liouville's Theorem:

A phase space volume / density does not change when it is transported along the beam line!

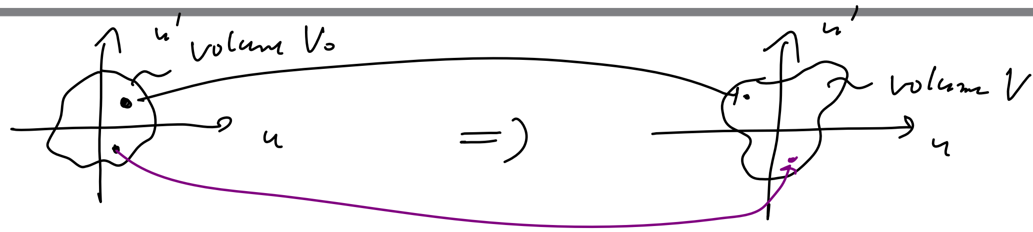
$\Rightarrow$ (u, u') phase space occupied by the beam

= area of phase space ellipse enclosing a certain fraction of particles of the beam

= emittance $\varepsilon \cdot \pi$ $\qquad \varepsilon = 2 J_{ellipse}$ here

= const ! $\qquad [\varepsilon] = m\,(rad)$

condition: Hamiltonian motion with constant energy (i.e. no acceleration, neglecting effects like beam scattering, synchrotron radiation, ...)



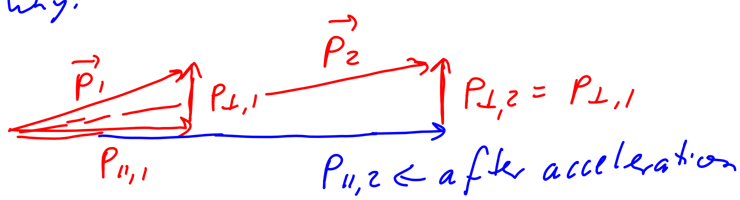
$$\underline{\underline{\text{Volume} = V}} = \iint_V d^n\vec{x} = \iint_{V_0} \left|\frac{\partial \vec{x}}{\partial \vec{x}_0}\right| d^n\vec{x}_0 = \iint_{V_0} |\underline{M}|\, d^n\vec{x}_0 = \iint_{V_0} d^n\vec{x}_0 = \underline{\underline{V_0}}$$

$$\vec{x} = \underline{M}(s)\,\vec{x}_0 \quad \text{with} \quad \det|\underline{M}| = 1$$

$$\boxed{V = V_0 \quad \Longleftrightarrow \quad \det \underline{M} = 1}$$

Note: If particles are accelerated, the emittance ε decreases inversely proportional to the momentum

why:



$$\Rightarrow \quad x_2' = \frac{dx_2}{ds} = \frac{p_{\perp,2}}{p_{0,2}} < x_1' = \frac{dx_1}{ds} = \frac{p_{\perp,1}}{p_{0,1}}$$

$$\Rightarrow \quad \varepsilon_2 < \varepsilon_1$$

recall: Hamiltonian formalism

Canonically conjugate momenta of position u is not u' but transverse momentum $p_u = p_0 u'$

$\Rightarrow$ phase space ellipse in (u, p_u) plane with constant area $= \pi \varepsilon p_0$

$= \pi \varepsilon_N (m_0 c)$

with normalized emittance:

$$\varepsilon_N = \left(\frac{p_0}{m_0 c}\right) \varepsilon = \text{const, even if accelerated}$$

$\beta = v/c$

$$= \beta \gamma \varepsilon$$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$\Rightarrow$ for $\beta \approx 1 \Rightarrow \varepsilon \propto \frac{1}{\gamma}$

$\Rightarrow$ ε_N stays constant during acceleration (as long as synchrotron radiation effects can be neglected!)

- for Gaussian particle distribution in phase space (often good fit for electrons)

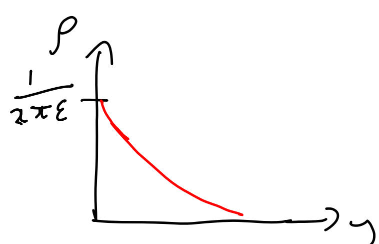
$$\rho(u,u') = \frac{1}{2\pi\varepsilon}\, e^{-\frac{\gamma u^2 + 2\alpha u u' + \beta u'^2}{2\varepsilon}}$$

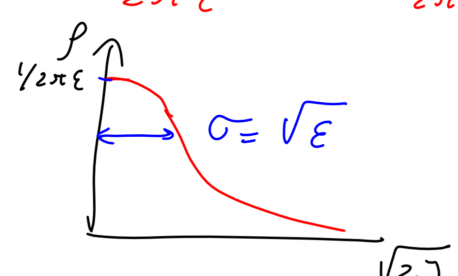
$\Rightarrow$ equi-density lines are ellipses in phase space

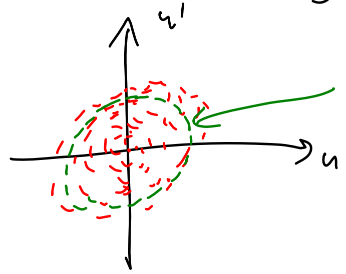
$\Rightarrow$ choose starting conditions for α, β, γ according to initial particle distribution

$\Rightarrow$ since $\gamma u^2 + 2\alpha u u' + \beta u'^2 = 2J$

$$\Rightarrow \rho(J, \phi_0) = \frac{1}{2\pi\varepsilon}\, e^{-J/\varepsilon} = \frac{1}{2\pi\varepsilon}\, e^{-\frac{(\sqrt{2J})^2}{2\varepsilon}}$$







phase space ellipse with $2J_{max} = \varepsilon$ contains

$$\int_0^{2\pi}\int_0^{J_{max}} \frac{1}{2\pi\varepsilon}\, e^{-J/\varepsilon}\, d\phi_0\, dJ = 1 - e^{-1/2} = 39\%$$

of all particles

⇒ since:

$$\langle 1 \rangle = \frac{1}{2\pi\varepsilon}\int_0^{2\pi}\int_0^{\infty} e^{-J/\varepsilon}\, dJ\, d\phi_0 = 1$$

$$\langle x^2 \rangle = \frac{1}{2\pi\varepsilon}\int_0^{2\pi}\int_0^{\infty} 2J\beta \sin^2\phi_0\, e^{-J/\varepsilon}\, dJ\, d\phi_0 = \varepsilon\beta$$

$$\langle x'^2 \rangle = \varepsilon\gamma$$

$$\langle x x' \rangle = -\frac{1}{2\pi\varepsilon}\int_0^{2\pi}\int_0^{\infty} 2J\alpha \sin^2\phi_0\, e^{-J/\varepsilon}\, dJ\, d\phi_0 = \varepsilon\alpha$$

gives

emittance $= \varepsilon = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2}$

$$= \varepsilon\sqrt{\beta\underbrace{\gamma}_{\gamma = \frac{1+\alpha^2}{\beta}} - \alpha^2} = \varepsilon\sqrt{1+\alpha^2-\alpha^2} = \underline{\underline{\varepsilon}}$$