Lecture 11

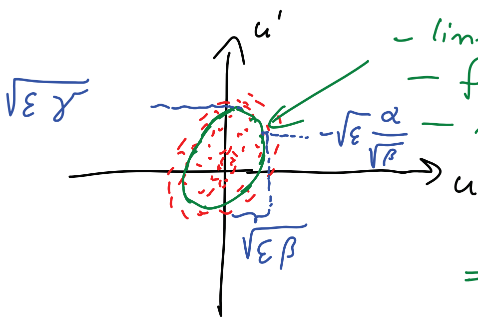
3. Linear transverse beam optics

3.8 Emittance and beam envelope

3.9 Propagation of the Twiss parameters

3.10 Transport matrix from the Twiss parameters

- Initial Twiss parameters from initial beam distribution (for non-periodic beam optics)



- line of constant phase space density
- fit by properly parametrized ellipses
- select ellipse which contains a certain fraction of particles (e.g. 39 %)

$\Rightarrow$ area of ellipse $= \pi\,\varepsilon$
$= \pi$ emittance

$\Rightarrow$ shape and orientation of ellipse give $\alpha_0, \beta_0\,(\gamma_0)$

beam envelope:

define "beam envelope phase space ellipse" by

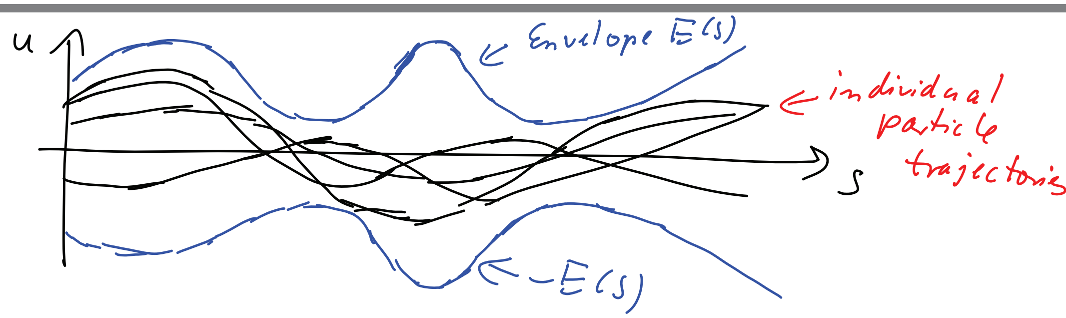
$$u(s) = \sqrt{\varepsilon\, \beta(s)}\, \sin(\psi(s) + \Phi_0)$$

$\Rightarrow$ position dependent beam envelope:

$$E(s) = \sqrt{\varepsilon\, \beta(s)}$$

i.e. β-function is a measure of the beam cross-section along the beam line

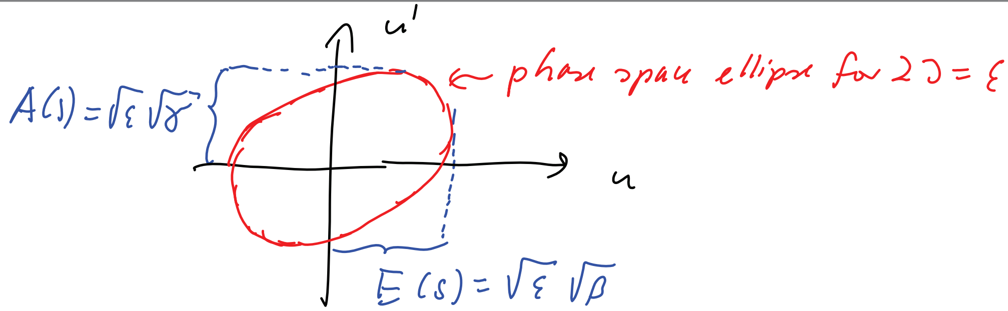
$\Rightarrow$ particles inside "beam envelope phase space ellipse" undergo betatron oscillations with particle trajectories inside $\pm E(s)$



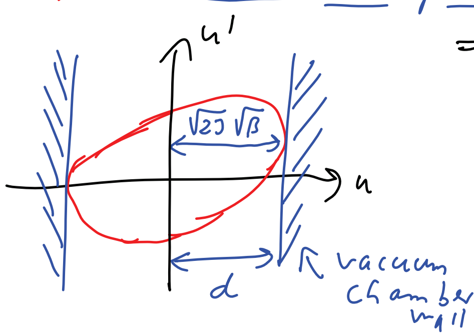
$\Rightarrow$ position dependent beam divergence:

$$A(s) = \frac{\sqrt{\varepsilon}}{\sqrt{\beta}} \sqrt{1+\alpha^2} = \sqrt{\varepsilon}\, \sqrt{\gamma}$$

i.e. the γ-function is a measure of beam divergence



⇒ Transverse acceptance of an accelerator



⇒ largest possible phase space ellipse:

$$\sqrt{2J_{max}}\sqrt{\beta_{max}} = d$$

$$\Rightarrow 2J_{max} = \text{acceptance} = \left(\frac{d^2}{\beta}\right)_{min} = N \cdot \varepsilon$$

⇒ Electron storage ring: need acceptance > 50 ε for beam lifetime

3.9 Propagation of Twiss Parameters

⤳ important to know how beam envelope $E = \sqrt{\varepsilon\beta}$ and beam divergence $A(s) = \sqrt{\varepsilon\gamma}$ change along beam line

⤳ need to know how to transform α, β, γ through the magnet structure along beam line, starting with initial $\alpha_0, \beta_0, \gamma_0$ at reference point s_0

⤳ "Methode 1":

define beta matrix:

$$B = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \Leftrightarrow B^{-1} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$$

note: $\det B = \beta\gamma - \alpha^2 = 1$

$\Rightarrow$ at reference point s_0 and an arbitrary other point s:

$$(u_0, u_0')\, \underline{B}_0^{-1} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} = (u_0, u_0') \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$

$$= \gamma_0 u_0^2 + 2\alpha_0 u_0 u_0' + \beta_0 u_0'^2 = 2J$$

= const through motion!

$$\Rightarrow \underbrace{(u_0, u_0')\, \underline{B}_0^{-1} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}}_{\text{at } s_0} = \underbrace{(u, u')\, \underline{B}^{-1} \begin{pmatrix} u \\ u' \end{pmatrix}}_{\text{at } s}$$

$$\Rightarrow \text{ since: } \begin{pmatrix} u \\ u' \end{pmatrix} = \underline{M} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} \Leftrightarrow (u, u') = \left(\underline{M} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} \right)^T$$

$$= (u_0\ u_0')\, \underline{M}^T$$

$$\Rightarrow (u_0, u_0')\, B_0^{-1} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} = (u_0, u_0')\, \underline{M}^T \underline{B}^{-1} \underline{M} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$

$$\Rightarrow \quad \underline{B}_0^{-1} = \underline{M}^T \underline{B}^{-1} \underline{M}$$

$$\Rightarrow \quad (\underline{M}^T)^{-1} \underline{B}_0^{-1} \underline{M}^{-1} = \underline{B}^{-1}$$

$$\Rightarrow \quad \text{since } (\underline{A}\,\underline{B}\,\underline{C})^{-1} = \underline{C}^{-1} \underline{B}^{-1} \underline{A}^{-1}$$

$$\Rightarrow \quad \boxed{\underline{B} = \underline{M}\ \underline{B}_0\ \underline{M}^T}$$

} transform beta matrix B_0 at point s_0 into beta matrix B at point s

transformation matrices from before

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \underline{M} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \underline{M}^T$$

$\rightarrow$ "Method" 2:

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \underline{M} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} \quad \text{with} \quad \underline{M} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\Rightarrow \quad M^{-1} \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} \quad \text{with} \quad M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$\Rightarrow$ so:

$$u_0 = S'u - S\,u'$$
$$u_0' = -C'u + C\,u'$$

now: with:

$$2J = \gamma u^2 + 2\alpha u u' + \beta u'^2$$
$$= \gamma_0 u_0^2 + 2\alpha_0 u_0 u_0' + \beta_0 u_0'^2$$
$$= \gamma_0 (S'u - Su')^2 + 2\alpha_0 (S'u - Su')(-C'u + Cu')$$
$$+ \beta_0 (-C'u + Cu')^2$$
$$= (C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0)\,u^2$$
$$+ 2(-CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0)\,u u'$$
$$+ (C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0)\,u'^2$$

$\Rightarrow$ comparing coefficients gives:

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

$\Rightarrow$ in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

$\Rightarrow$ transform beta function piecewise through magnets and drifts

- Example: Twiss Parameters in a Drift:

$$M_{drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

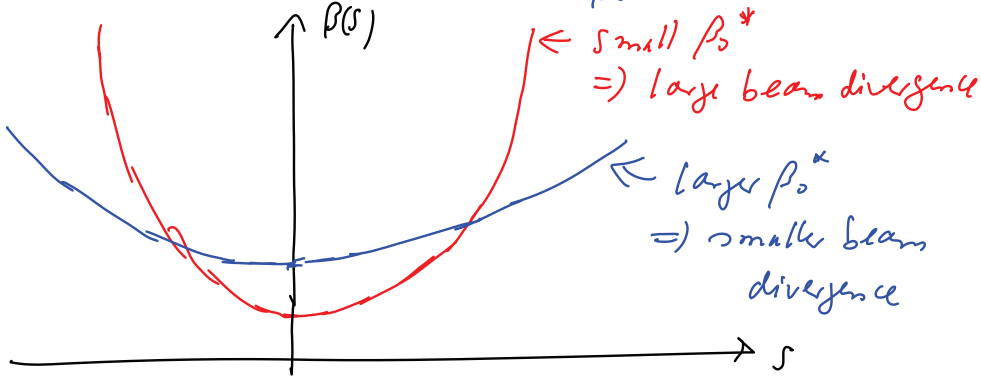
$$\Rightarrow \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & -2s & s^2 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

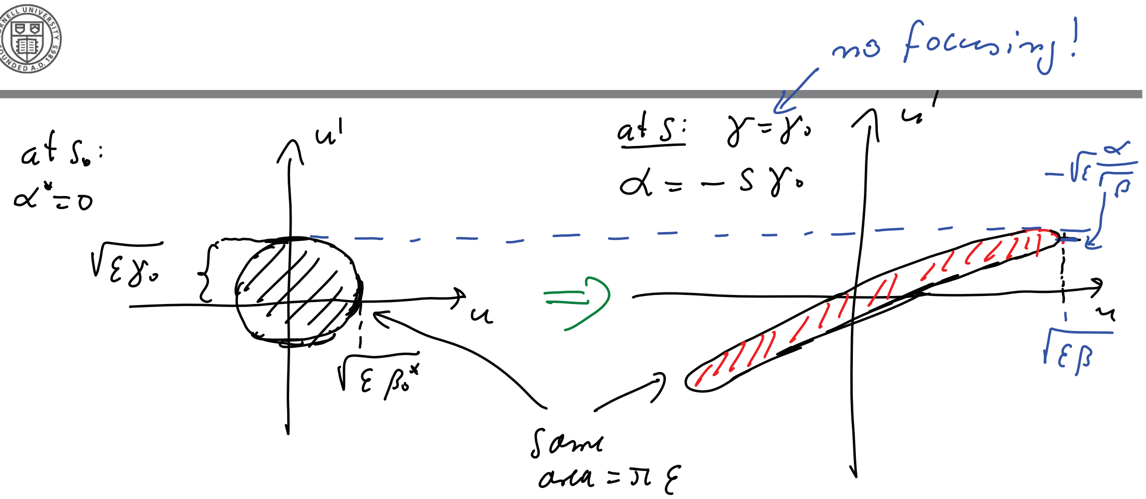
$$\Rightarrow \quad \beta(s) = \beta_0 - 2s\alpha_0 + s^2\gamma_0$$

⇒ about symmetry point in drift region at s_0,

i.e. for $\alpha_0 = 0$: $\beta_0 = \beta_0^*$

$$\beta(s) = \beta_0^* + \frac{s^2}{\beta_0^*}$$





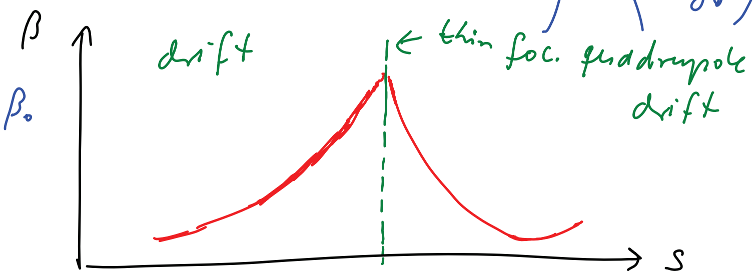
- Example: Twiss parameter in a thin quadrupole:

$$M_{\substack{\text{thin} \\ \text{foc. quad}}} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k^2 & 2k & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

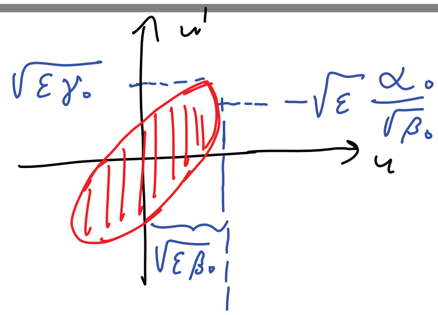
$\alpha = \alpha_0 + k\beta_0$

$= -\frac{1}{2}\beta'$



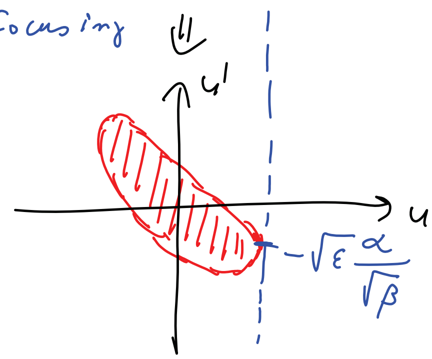
just before quad:

$\alpha_0 < 0$ here



just after quad:

focusing ⇓



$\alpha > 0$ here

$\beta \simeq \beta_0$ here