Lecture 14

4. Beam optics in circular accelerators

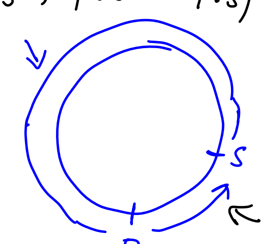
4.6 Closed orbit for $\Delta p \neq 0$

4.7 Effect of dipole kicks

4.8 Orbit correction and orbit bumps

$$u_{orb}(s) = \frac{\sqrt{\beta(s)}}{2\sin\left(\frac{\mu}{2}\right)} \oint_{\text{full turn}} p(\tilde{s})\sqrt{\beta(\tilde{s})}\cos\left\{|\psi(\tilde{s})-\psi(s)|-\frac{\mu}{2}\right\} d\tilde{s}$$

$\tilde{s} > s \Rightarrow \psi(\tilde{s}) > \psi(s)$



$\Leftarrow \tilde{s} < s$

$\psi(\tilde{s}) < \psi(s)$

$\psi(0) = 0$

$\psi(L) = \mu$

$$\cos\left(\psi(\tilde{s}) - \psi(s) - \frac{\mu}{2} + \mu\right\}$$

need to add factor μ if $\tilde{s}$ is smaller than s

$$= \cos\left\{\underbrace{\psi(s)-\psi(\tilde{s})}_{>0} - \mu + \frac{\mu}{2}\right\}$$

$$= \cos\left\{\underbrace{|\psi(\tilde{s})-\psi(s)| - \frac{\mu}{2}}_{\text{correct for any } \tilde{s}}\right\}$$

$\rightsquigarrow$ alternative way to derive $u_{orb}(s)$

$$u'' = -\mathcal{K} u + p(s) \leftarrow \text{consider extra kick at } s = \tilde{s} \text{ on beam}$$

$$\Delta u' = p(s)\,\Delta s = \underbrace{\Delta\vartheta}_{\text{kick angle}}$$

$$\Rightarrow \quad \vec{u}(s) = \underline{M}\,\vec{u}_0 + \underline{M}(\tilde{s} \to s)\begin{pmatrix} 0 \\ \Delta\vartheta \end{pmatrix}$$

$\Rightarrow$ for multiple kicks:

$$\vec{u}(s) = \underline{M}\,\vec{u}_0 + \sum_k \underline{M}(\tilde{s}_k \to s)\begin{pmatrix} 0 \\ \Delta\vartheta_k \end{pmatrix}$$

$\Rightarrow$ change $\sum$ to $\int$ for $p(s)$ error:

$$\vec{u}(s) = \underline{M}\,\vec{u}_0 + \int_0^s \underline{M}(\tilde{s} \to s)\begin{pmatrix} 0 \\ p(\tilde{s}) \end{pmatrix} d\tilde{s}$$

$\Rightarrow$ for periodic / closed orbit: $\vec{u}_{orb}(0) = \vec{u}_{orb}(L)$

$$\vec{u}_{orb}(0) = \underline{M}_p \vec{u}_{orb}(0) + \int_0^L \underline{M}(\tilde{s} \to L)\begin{pmatrix} 0 \\ p(\tilde{s}) \end{pmatrix} d\tilde{s}$$

$\underline{M}$ for full turn

$$= \underline{M}_p \vec{u}_{orb}(0) + \int_0^L \begin{pmatrix} \sqrt{\beta_0 \beta(\tilde{s})}\sin(\psi(L) - \psi(\tilde{s})) \\ \sqrt{\frac{\beta(\tilde{s})}{\beta_0}}\left[\cos(\psi(L) - \psi(\tilde{s})) - \alpha_0 \sin(\psi(L) - \psi(\tilde{s}))\right] \end{pmatrix} \cdot p(\tilde{s})\, d\tilde{s}$$

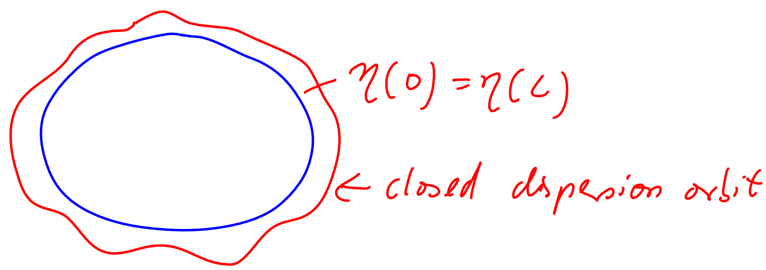
$\Rightarrow$ solve for $\vec{u}_{orb}(0)$ ....

4.6 Closed orbit for $\Delta p \neq 0$

$$\Rightarrow \quad p(\hat{s}) = \frac{1}{\rho(\hat{s})}\,\delta$$

$\Rightarrow$ closed periodic dispersion function : $x_D = D\,\delta$

$$\boxed{\eta(s) \equiv D_{\text{periodic}}(s) = \frac{\sqrt{\beta(s)}}{2\sin(\mu/2)} \oint \frac{\sqrt{\beta(\hat{s})}}{\rho(\hat{s})} \cos\left(|\psi(\hat{s}) - \psi(s)| - \frac{\mu}{2}\right) d\hat{s}}$$



$\Rightarrow$ particle trajectory:

$$\vec{x} = \underbrace{\underline{M}_2\,\vec{x}_{\beta,0}}_{\substack{\text{betatron}\\ \text{oscillation}}} + \underbrace{\vec{\eta}(s)\,\delta}_{\text{for dispersion}} \qquad \text{with } \vec{\eta} = \begin{pmatrix} \eta \\ \eta' \end{pmatrix}$$

note: $\vec{x}_0 = \vec{x}_{\beta,0} + \underbrace{\vec{\eta}(0)\,\delta}_{\neq 0}$!

- η : periodic dispersion function with $\eta(0) = \eta(L)$ and $\eta(0) \neq 0$

$D(s)$ dispersion function with $D(0)=0$

$$\Rightarrow \quad \vec{\eta}(L) = \begin{pmatrix} \eta \\ \eta' \end{pmatrix}_L = \underline{M}_{2,\text{ full turn}, x}\, \vec{\eta}(0) + \vec{D}(L)$$

$$\Rightarrow \quad \vec{\eta}(0) = \underline{M}_{2,\text{ full turn}, x}\, \vec{\eta}(0) + \vec{D}(L)$$

$$\Rightarrow \quad \boxed{\vec{\eta}(0) = \left[1 - \underline{M}_{2,\text{ full turn}, x} \right]^{-1} \vec{D}(L)}$$

$$\eta(0) = \frac{(1-S')D + SD'}{2 - C - S'} = \frac{(1-S')D + SD'}{4\sin^2(\mu/2)}$$

$$\eta'(0) = \frac{C'D + (1-C)D'}{4\sin^2(\mu/2)}$$

- To calculate $\eta(s)$ at any other point s from initial $\eta(0)$ and $\eta'(0)$:

$$\boxed{\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_s = \underline{M}_{3,x}(s_0 \to s) \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_{s_0}}$$

4.7 Effect of dipole field error kicks

⤳ assume dipole field error of strength ΔB, acting over length ℓ

$\Rightarrow$ kick angle: $\Delta x' = \Delta \vartheta = \frac{q}{p} \Delta B \, \ell$

⤳ in following: assume kicks are localized at $s = s_k$
$k = 1, 2, \ldots$

$\Rightarrow$ resulting distorted orbit (from above)

$$u_{orb}(s) = \sum_k \Delta \vartheta_k \frac{\sqrt{\beta(s)\,\beta(s_k)}}{2 \sin(\mu/2)} \cos\left(|\psi(s_k) - \psi(s)| - \frac{\mu}{2}\right)$$

effect of field error increases with beta function at point of disturbance!

instability when tune $\nu = \mu/2\pi$ is integer

- for single localized kick:

$$u_{orb}(s) = \Delta \vartheta_k \frac{\sqrt{\beta(s)\,\beta(s_k)}}{2 \sin(\mu/2)} \cos\left(|\psi(s_k) - \psi(s)| - \frac{\mu}{2}\right)$$

$\Rightarrow$ for $s > s_k$: $\psi > \psi_k$ $\Rightarrow$ free betatron oscillation!

$$u_{orb}(s) = \Delta \vartheta_k \frac{\sqrt{\beta(s)\,\beta(s_k)}}{2 \sin(\mu/2)} \cos\left(\psi(s) - \psi(s_k) - \mu/2\right)$$

$$\stackrel{!}{=} \sqrt{2J\,\beta(s)} \sin(\psi + \phi_0)$$

at s_0

gives: $J = \Delta \vartheta_k^2 \frac{\beta(s_k)}{8 \sin^2(\mu/2)}$ $\qquad \phi_0 = \frac{\pi}{2} - \psi(s_k) - \mu/2$

$\Rightarrow$ for $s < s_k$: $\psi < \psi_k \Rightarrow$ free betatron oscillation

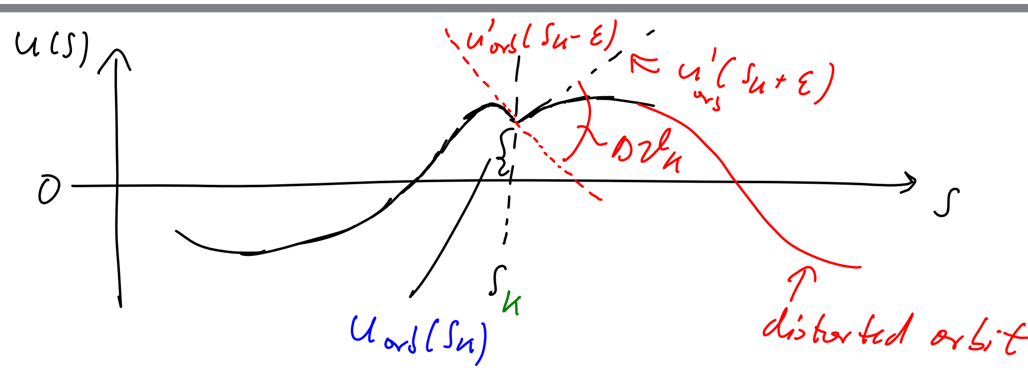
$$u_{orb}(s) = \Delta \vartheta_k \frac{\sqrt{\beta(s)\beta(s_k)}}{2\sin(\mu/2)} \cos\left(\psi(s) - \psi(s_k) + \mu/2\right)$$

$$= \sqrt{2 J \beta(s)} \sin(\psi - \psi_0)$$

gives: $J = \Delta \vartheta_k^2 \frac{\beta(s_k)}{8\sin^2(\mu/2)}$ $\quad \phi_0 = \frac{\pi}{2} - \psi(s_k) + \frac{\mu}{2}$

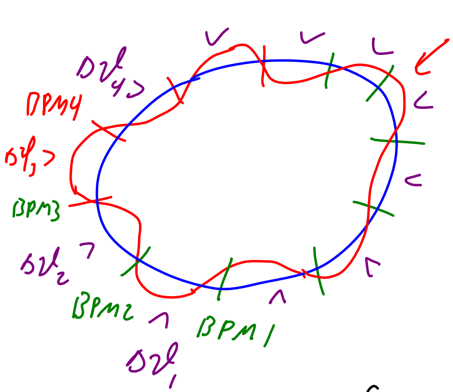
$\Rightarrow$ at s_k:

$$u_{orb}(s_k) = \Delta \vartheta_k \frac{\beta(s_k)}{2\tan(\mu/2)} \qquad u'_{orb}(s_k+\varepsilon) = \frac{\Delta\vartheta_k}{2}\left(1 - \frac{\alpha(s_k)}{\tan(\mu/2)}\right)$$



4.8 Closed Orbit correction and orbit bumps

orbit correction:



distorted closed orbit

→ beam position monitors $m = 1 \dots n$ measure distorted orbit

$$u_{orb}(s_m)$$

→ use corrector magnets $k = 1 \dots N$ to correct orbit

⇒ before adjusting the orbit: $u_{orb}^{old}(s_m)$

⇒ after adjusting the orbit with the correctors:

$$u_{orb}^{new}(s_m) = u_{orb}^{old}(s_m) + \sum_k \Delta\vartheta_k \frac{\sqrt{\beta(s_m)\beta(s_k)}}{2\sin(\mu/2)} \cdot \cos\left(|\psi(s_k) - \psi(s_m)| - \frac{\mu}{2}\right)$$

$$= u_{orb}^{old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$

$$\rightarrow \vec{u}_{orb}^{\,new} = \begin{pmatrix} u_{orb}^{new}(s_1) \\ \vdots \\ u_{orb}^{new}(s_n) \end{pmatrix} = \vec{u}_{orb}^{\,old} + \underline{O}\,\Delta\vec{\vartheta}$$

$\Rightarrow$ for perfect orbit: want $\vec{u}_{obs}^{\,new} = \vec{0}$

adjust correctors: $\boxed{\Delta\vec{\vartheta} = -\underline{O}^{-1}\,\vec{u}_{obs}^{\,old}}$

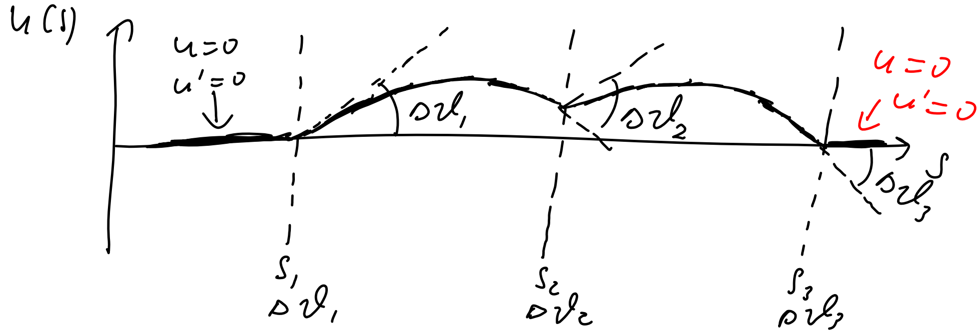
Orbit bumps:

→ often necessary to shift beam within limited region (e.g. during injection) → local orbit bumps

→ without affecting rest of ring!

→ use sequence of small dipole kick magnets

Example: with three correcting coils:



① $u_{orb}(s_1) = 0 = \sum_{k=1}^{3} \Delta\vartheta_k \frac{\sqrt{\beta(s_1)\beta(s_k)}}{2\sin(\mu/2)} \cos(|\psi(s_k)-\psi(s_1)|-\frac{\mu}{2})$

② $u_{orb}(s_3) = 0 = \sum_{k=1}^{3} \Delta\vartheta_k \frac{\sqrt{\beta(s_3)\beta(s_k)}}{2\sin(\mu/2)} \cos(|\psi(s_k)-\psi(s_3)|-\frac{\mu}{2})$

from ①:

$$\frac{\Delta\vartheta_1}{\Delta\vartheta_2}\sqrt{\beta(s_1)}\cos\left(\frac{\mu}{2}\right) + \frac{\Delta\vartheta_3}{\Delta\vartheta_2}\sqrt{\beta(s_3)}\cos\left(\psi(s_3)-\psi(s_1)-\mu/2\right)$$

$$= -\sqrt{\beta(s_2)}\cos\left(\psi(s_2)-\psi(s_1)-\mu/2\right)$$

from ②:

$$\frac{\Delta\vartheta_1}{\Delta\vartheta_2}\sqrt{\beta(s_1)}\cos\left(|\psi(s_1)-\psi(s_3)|-\frac{\mu}{2}\right) + \frac{\Delta\vartheta_3}{\Delta\vartheta_2}\sqrt{\beta(s_3)}\cos\left(\mu/2\right)$$

$$= -\sqrt{\beta(s_2)}\cos\left(|\psi(s_2)-\psi(s_3)|-\mu/2\right)$$

$\Rightarrow$

$$\begin{pmatrix} \frac{\Delta\vartheta_1}{\Delta\vartheta_2} \\ \frac{\Delta\vartheta_3}{\Delta\vartheta_2} \end{pmatrix} = -\frac{\sqrt{\beta(s_2)}}{N} \begin{pmatrix} \sqrt{\frac{1}{\beta(s_1)}}\cos(\mu/2) & -\sqrt{\frac{1}{\beta(s_1)}}\cos(\psi_3-\psi_1-\mu/2) \\ -\sqrt{\frac{1}{\beta(s_3)}}\cos(\psi_3-\psi_1-\mu/2) & \sqrt{\frac{1}{\beta(s_3)}}\cos(\mu/2) \end{pmatrix} \cdot \begin{pmatrix} \cos(\psi_2-\psi_1-\mu/2) \\ \cos(\psi_3-\psi_2-\mu/2) \end{pmatrix}$$

with $\mathcal{N} = \cos^2(\mu/2) - \cos^2(\psi_3 - \psi_1 - \mu/2)$

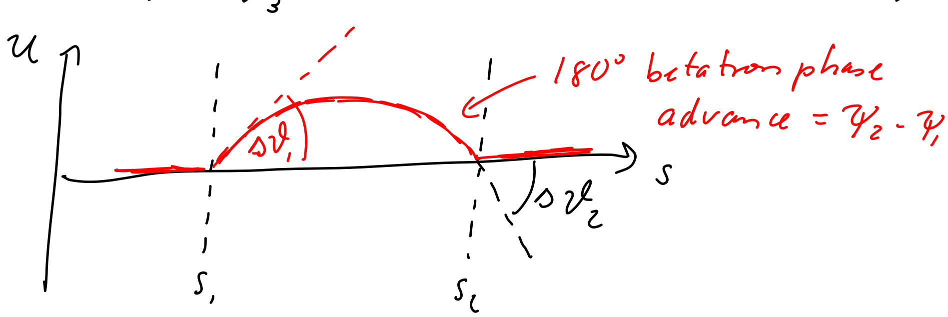
$$= \sin(\psi_3 - \psi_1 - \mu)\,\sin(\psi_3 - \psi_1)$$

$$\Rightarrow \begin{pmatrix} \frac{\Delta\vartheta_1}{\Delta\vartheta_2} \\ \frac{\Delta\vartheta_3}{\Delta\vartheta_2} \end{pmatrix} = -\frac{1}{\mathcal{N}} \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}\,\sin(\psi_3 - \psi_1 - \mu)\,\sin(\psi_3 - \psi_2) \\ \sqrt{\frac{\beta(s_2)}{\beta(s_3)}}\,\sin(\psi_3 - \psi_1 - \mu)\,\sin(\psi_2 - \psi_1) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{\Delta\vartheta_1}{\Delta\vartheta_2} \\ \frac{\Delta\vartheta_3}{\Delta\vartheta_2} \end{pmatrix} = -\frac{1}{\sin(\psi_3 - \psi_1)} \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}\,\sin(\psi_3 - \psi_2) \\ -\sqrt{\frac{\beta(s_2)}{\beta(s_3)}}\,\sin(\psi_2 - \psi_1) \end{pmatrix}$$

Note: for $\psi_2 - \psi_1 = 180° \cdot \text{integer}$

$\Rightarrow \Delta\vartheta_3 = 0 \quad \Rightarrow$ need two correctors only



$$\Rightarrow \frac{\Delta\vartheta_1}{\Delta\vartheta_2} = \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}$$