Lecture 15

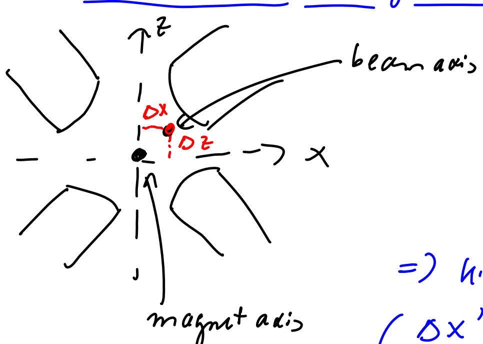
4. Beam optics in circular accelerators

4.9 Quadrupole errors

4.10 Chromaticity and its correction

4.9 Quadrupole error

① Transverse misalignment



at orbit

$$\begin{pmatrix} \Delta B_x \\ \Delta B_z \end{pmatrix} = g \begin{pmatrix} \Delta z \\ \Delta x \end{pmatrix}$$

⇒ kick in both planes (angular deflection)

$$\begin{pmatrix} \Delta x' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \Delta \vartheta_x \\ \Delta \vartheta_z \end{pmatrix} = \frac{q}{p} g \ell \begin{pmatrix} \Delta x \\ \Delta z \end{pmatrix} = k\ell \begin{pmatrix} \Delta x \\ \Delta z \end{pmatrix}$$

↑ length of quadrupole

⇒ orbit distortion, like for dipole errors:

$$x_{orb}(s) = \Delta\vartheta_k \frac{\sqrt{\beta(s)\beta(s_k)}}{2\sin(\mu/2)} \cos\left\{|\psi(s_k) - \psi(s)| - \frac{\mu}{2}\right\} \propto \sqrt{\beta(s_k)}\, k\ell\, \Delta x$$

2 Quadrupole gradient error

small gradient error: $g = g_0 + \Delta g$ $\quad |\Delta g| \ll |g|$

$\Rightarrow$ quadrupole strength error: $k = k_0 + \Delta k$

$\Rightarrow$ change in focusing $\Rightarrow$ change in tune of accelerator

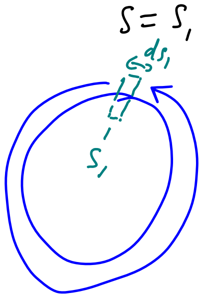
$\Rightarrow$ change in β-function

2a) Change in tune:

- transformation matrix for full revolution in the ideal machine:

$$\underline{M}_0 = I \cos\mu_0 + J \sin\mu_0 \quad , \mu_0 = 2\pi\nu_0$$

- suppose quad gradient error occurs only at $s = s_1$ over a short length ds_1:



$\Rightarrow$ transformation matrix for disturbed machine from s_1 to $s_1 + L$

$$\underline{M} = \underbrace{\underline{m}}_{\substack{\text{matrix for section of length} \\ ds_1 \text{ in the disturbed case}}} \; \underbrace{\underline{m}_0^{-1}}_{\substack{\text{matrix of section of length} \\ ds_1 \text{ in the ideal case}}} M_0$$

neglect short length of section

So:

$$m_0 = \begin{pmatrix} 1 & 0 \\ -K_0(s_1)ds_1 & 1 \end{pmatrix}$$

$$m = \begin{pmatrix} 1 & 0 \\ -(K_0(s_1)+\Delta k(s_1))ds_1 & 1 \end{pmatrix}$$

$\Rightarrow$ to first order in ds_1

$$M = \begin{pmatrix} 1 & 0 \\ -\Delta k\, ds_1 & 1 \end{pmatrix} \cos\mu_0 + \begin{pmatrix} \alpha & \beta \\ -\alpha \Delta k\, ds_1 - \gamma & -\beta \Delta k\, ds_1 - \alpha \end{pmatrix} \sin\mu_0$$

$\Rightarrow$ for betatron phase advance:

$$\cos\mu = \frac{1}{2}\,\mathrm{trace}\, M = \cos\mu_0 - \frac{1}{2}\beta(s_1)\Delta k(s_1)\sin\mu_0\, ds_1$$

$\Rightarrow$ for small $\Delta\mu$

$$\Delta(\cos\mu) = \cos\mu - \cos\mu_0 = -\frac{1}{2}\beta(s_1)\Delta k(s_1)\sin\mu_0\, ds_1$$

$$\approx \frac{d\cos\mu}{d\mu}\Delta\mu = -\sin\mu_0\,\Delta\mu$$

$\Rightarrow$ change in tune due to a single quadrupole gradient error

$$\Delta\nu = \frac{\Delta\mu}{2\pi} = \frac{1}{4\pi}\beta(s_1)\Delta k(s_1)\,ds_1$$

⇒ more focusing always increases tune

⇒ $\Delta\nu \propto \Delta K$, $\Delta\nu \propto \beta(s_1)$

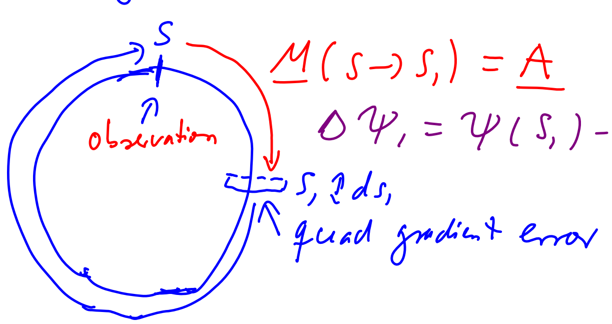
⇒ measurement of β-function: change K and measure change in tune (oscillation frequencies can be measured easily and accurately!)

⇒ for quad gradient error distributed along the ring:

tune shift: $$\boxed{\Delta\nu = \frac{1}{4\pi}\oint \beta(s)\,\Delta k(s)\,ds}$$

Note: formula only valid for small Δk!

2b) Change in beta-function ("beta-beat")



$\underline{M}(s \to s_1) = \underline{A}$

$\Delta\psi_1 = \psi(s_1) - \psi(s)$

$\underline{M}(s_1 \to s) = \underline{B}$

$\Delta\psi_2 = \mu - \Delta\psi_1$

- unperturbed:

$$\underline{M}_{rev}(s) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \underline{M}(s_1 \to s)\, \underline{M}(s \to s_1)$$
$$= \underline{B} \cdot \underline{A}$$

with $m_{12} = \beta(s) \sin\mu = b_{11} a_{12} + b_{12} a_{22}$

$$a_{12} = \sqrt{\beta(s)\beta(s_1)} \sin\{\psi(s_1) - \psi(s)\}$$

$$b_{12} = \sqrt{\beta(s)\beta(s_1)} \sin\{\mu - (\psi(s_1) - \psi(s))\}$$

- perturbed:

$$\underline{M}^*_{rev}(s) = \begin{pmatrix} m_{11}^* & m_{12}^* \\ m_{21}^* & m_{22}^* \end{pmatrix} = \underline{B} \cdot \begin{pmatrix} 1 & 0 \\ -\Delta k\, ds_1 & 1 \end{pmatrix} \cdot \underline{A}$$

with: $m_{12}^* = (\beta(s) + \underline{\underline{\Delta\beta(s)}}) \sin\{2\pi(\nu + \Delta\nu)\}$

$$= \underbrace{b_{11} a_{12} + b_{12} a_{22}}_{m_{12}} - a_{12} b_{12} \Delta k\, ds_1$$

$$= \beta(s) \sin\mu - a_{12} b_{12} \Delta k(s_1)\, ds_1$$

$\Rightarrow$ for small $\Delta\nu$, $\Delta\beta$ and to first order:

with $\cos(2\pi\Delta\nu) \simeq 1 \qquad \sin(2\pi\Delta\nu) \simeq 2\pi\Delta\nu$

$$(\beta(s) + \Delta\beta(s))\{\sin\mu + 2\pi\Delta\nu\cos\mu\} = \beta(s)\sin\mu - a_{12}b_{12}\Delta k \cdot ds_1$$

$\Rightarrow$ with a_{12}, b_{12}, $\Delta\nu$ from above:

$$\Delta\beta(s) = -\frac{\beta(s)\,\beta(s_1)}{2\sin\mu}\Big\{\underbrace{2\sin(\psi(s_1)-\psi(s))\sin(\mu-(\psi(s_1)-\psi(s))) + \cos\mu}_{=\cos(2(\psi(s_1)-\psi(s))-\mu)}\Big\}\Delta k(s_1)\,ds_1$$

$\Rightarrow$ for distributed quadrupole gradient error:

$\int$ over ds_1

$$\Delta\beta(s) = -\frac{\beta(s)}{2\sin\mu}\oint \beta(\tilde{s})\,\Delta k(\tilde{s})\cos\left[2|\psi(\tilde{s})-\psi(s)|-\mu\right]d\tilde{s}$$

Note: $-\Delta\beta \propto \beta(s_1) = \beta$-function at point of perturbation

$\Rightarrow$ gradient errors in regions with large β-function most dangerous (e.g. in the interaction region quadrupoles)

$-$ instability from quadrupole gradient errors for half-integer tunes! i.e. for $2\nu = $ integer

- formula only valid for small Δk!
- beta function beat $\Delta\beta$ oscillats twice as fast as orbit
  - extra focusing can increase or decrease beta function
  - $$\frac{\beta_{max}}{\beta} = 2\pi \frac{\Delta\nu}{\sin\mu}$$ } for one quadrupole error

4.10 Chromaticity and its correction

particle beam has spread in momentum ($\delta \simeq 10^{-3}$)

<u>but:</u> focusing in quadrupoles depends on particle momentum! $\frac{1}{f} = k\ell = \frac{q}{p} g \ell$

from: $k = \frac{q g}{p}$

one gets: $$\Delta k = \frac{dk}{dp} \Delta p = -\frac{q g}{p_0} \frac{\Delta p}{p_0} = -k\delta$$

$\Rightarrow$ quadrupole gradient error, which depends on momentum error δ

$\Rightarrow$ gives spread in tune : tune shift

$$\Delta\nu = -\frac{1}{4\pi}\oint \beta(s)\, k(s)\, ds \cdot \delta$$

$\Rightarrow$ define chromaticity ξ = energy dependence of tune

$$\nu(\delta) = \nu_0 + \frac{\partial \nu}{\partial \delta}\delta + \dots$$

$$= \nu_0 + \xi\delta$$

$\Rightarrow$ natural chromaticity ξ_0 from energy dependence of the tune from quadrupoles only:

$$\boxed{\xi_0 = \frac{\Delta\nu}{\delta} = -\frac{1}{4\pi}\oint \beta(s)\, k(s)\, ds}$$

Note: – large contribution from strong quadrupoles in regions with large β (e.g. interaction region)

– for linear magnet lattice (dipoles and quads only)

$$\xi < 0$$

why? $\beta > 0$ always

$$\beta_{foc.\,quad} > \beta_{def.\,quad}$$

$$k_{foc} > 0 \qquad k_{defoc} < 0$$

– natural chromaticity in large storage rings can be quite significant

Example: $\xi_{HERA,0} \approx -60 \Rightarrow$ large spread in tune!

for $\pm 10^{-3}$ relative momentum spread $\Rightarrow$ tune range ± 0.06

$\Rightarrow$ part of particles in the beam hit resonances at tunes (ν_x, ν_z) with instability $\Rightarrow$ beam loss

$\Rightarrow$ also: "head-tail" instability for $\xi < 0$

$\Rightarrow$ would like to adjust tune to slightly positive values: $\xi_{total} \approx +1 \ldots +3$