Lecture 16

4. Beam optics in circular accelerators

4.10 Chromaticity and its correction

4.11 Restriction of the dynamic aperture by sextupoles

4.12 Optical resonances

Chromaticity

spread in momentum ⇒ quadrupole gradient error

⇒ spread in tune

Chromaticity (natural)

$$\xi = \frac{\Delta \nu}{\delta} = -\frac{1}{4\pi} \oint \beta(s) K(s)\, ds < 0$$

⇒ can cause particle loss

⇒ need compensations

• Chromaticity compensation with sextupoles

~> place sextupole at location where closed orbit dispersion $\eta(s)$ is non zero

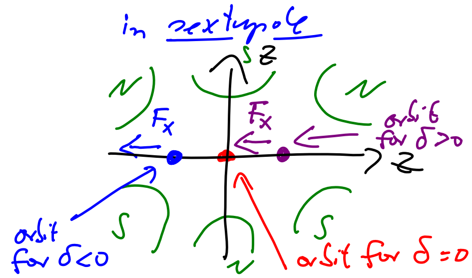
~> $x_D(s) = \eta(s)\,\delta$ in sextupole

~> energy dependent offset in sextupole leads to energy dependent quadrupole effect

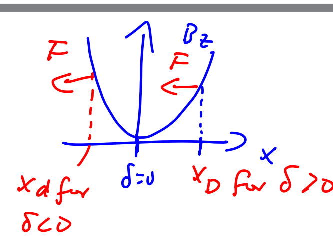
Example: in focusing quadrupole

if $\delta > 0$: too weak focusing

if $\delta < 0$: too strong focusing



• for $z = 0$: $B_z = \frac{1}{2} g' x^2$



• for $\delta > 0$ $\quad x = x_D + x_\beta \quad$ with $\quad x_D = \eta\,\delta$

$z = z_\beta$

=> for small quantities in x_β, z_β:

$$B_x = g' x z \approx g' x_D z_\beta = (g' \eta(s)\,\delta)\, z_\beta$$

$$B_z = \frac{1}{2} g'(x^2 - z^2) \approx \underbrace{\frac{1}{2} g' x_D^2}_{\text{causes deflection of closed dispersion orbit}} + (g' \eta(s)\,\delta)\, x_\beta$$

⇒ sextupole acts like quadrupole with equivalent quadrupole strength

$$\Delta k_{sext} = \frac{q}{p_0} g' \eta(s) \delta = m\, \eta(s)\, \delta$$

$m = \frac{q}{p_0} g'$: sextupole strength

note: – opposite effect in horizontal vs. vertical plane!

– want $\Delta k > 0$ to increase tune for $\delta > 0$ case

⇒ total chromaticity of ring with quadrupoles and sextupoles:

$$\boxed{\xi = -\frac{1}{4\pi} \oint \left[k(s) - \underbrace{m(s)\, \eta(s)}_{\text{adjust for compensation}} \right] \beta(s)\, ds}$$

– place sextupole where β is large (i.e. next to focusing quadrupole)

– correct sign:

$$\xi_x = -\frac{1}{4\pi} \oint \left[k - m\, \eta_x \right] \beta_x\, ds$$

$$\xi_z = +\frac{1}{4\pi} \oint \left[k - m\, \eta_x \right] \beta_z\, ds$$

horiz. focusing quadrupole : $k > 0$

but: – sextupole introduce non-linear fields and coupling between the horizontal and vertical motion

=> non-harmonic betatron motion

=> frequency of oscillation, tune depend on amplitude of oscillation

=> non-linear resonances, sudden particle loss above certain amplitude (chaotic particle motion)

4.11 Restriction of dynamic aperture by sextupoles

sextupoles => non-linear particle dynamics

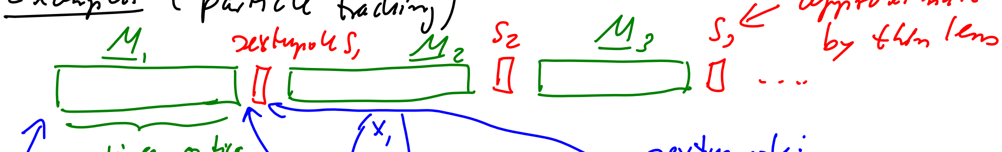
=> particle loss for large betatron amplitudes

=> limit in effective ("useable") aperture

=> "dynamic aperture"

=> study with numerical methods

Example: (particle tracking)



$$\vec{X}_0 = \begin{pmatrix} x_0 \\ x_0' \\ z_0 \\ z_0' \\ \tau_0 \\ \delta_0 \end{pmatrix} \leftarrow \text{initial phase space vector} \qquad \vec{X}_1 = \begin{pmatrix} x_1 \\ x_1' \\ z_1 \\ z_1' \\ \tau_1 \\ \delta_1 \end{pmatrix} = \underline{M_1} \vec{X}_0$$

sextupole:

$$\vec{B} = \begin{pmatrix} g' x_1 z_1 \\ \frac{1}{2} g' (x_1^2 - z_1^2) \\ 0 \end{pmatrix}$$

$\Rightarrow$ in sextupole: $\vec{F} = q\,\vec{v} \times \vec{B}$ over effective length of sextupole ℓ

$\Rightarrow$ kick:

$$\Delta x_1' = -\frac{q}{p} B_z \ell = -\frac{1}{2}(m\ell)(x_1^2 - z_1^2)$$

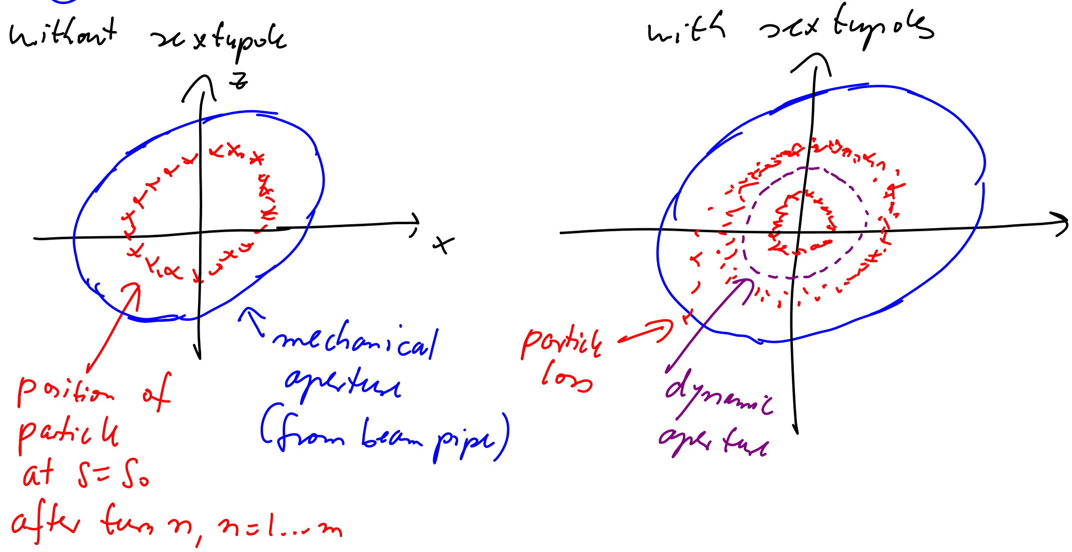
$$\Delta z_1' = \frac{q}{p} B_x \ell = (m\ell)\, x_1 z_1$$

$\Rightarrow$ just after sextupole:

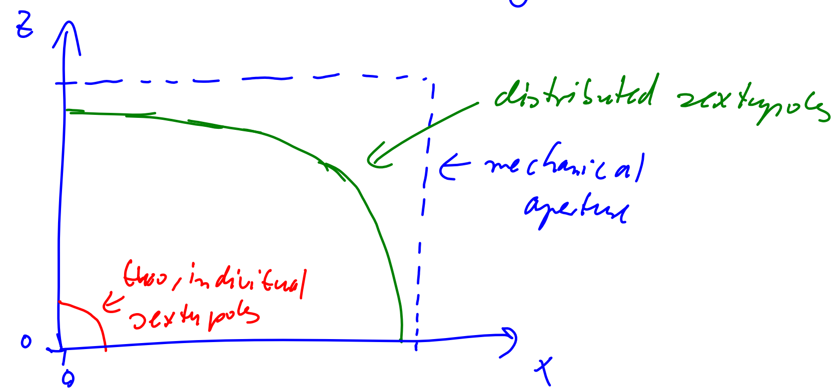
$$\vec{X}_2 = \begin{pmatrix} x_1 \\ x_1' + \Delta x_1' \\ z \\ z_1' + \Delta z_1' \\ \tau_1 \\ \delta \end{pmatrix}$$

Results:

① Particles with too large initial offset are lost:



(2) Chromaticity correction with large number of weak, distributed sextupoles gives larger dynamic aperture than with few, strong sextupoles!



4.12 Optical resonances

Step 1) Floquet's transformation of the homogeneous equation of motion: $u'' + K u = 0$

define:

$$\eta(s) \equiv \frac{u(s)}{\sqrt{\beta(s)}}$$ normalized particle trajectory amplitude

$$\phi \equiv \frac{\psi(s)}{\nu} = \frac{1}{\nu}\int \frac{1}{\beta(s)}\,ds$$ changes by 2π per turn

derivatives:

$$\frac{d\eta}{d\phi} = \frac{d\eta}{ds}\frac{ds}{d\phi} = \frac{d}{ds}\left(\frac{x(s)}{\sqrt{\beta(s)}}\right)\nu\beta(s)$$

$$= \left(\frac{\alpha(s)}{\sqrt{\beta(s)}}x(s) + \sqrt{\beta(s)}\,x'(s)\right)\nu$$

$$\frac{d^2\eta}{d\phi^2} = \frac{d}{d\phi}\left(\frac{d\eta}{d\phi}\right) = \frac{d}{ds}\left(\frac{d\eta}{d\phi}\right)\nu\beta(s)$$

$$= \left\{\beta^{3/2}x'' + \frac{\alpha^2}{\sqrt{\beta}}x + \underset{\substack{\uparrow \\ \alpha' = \beta\mathcal{K}(s) - \gamma}}{\alpha'}\sqrt{\beta}\,x\right\}\nu^2$$

$$\Rightarrow \frac{d^2\eta}{d\phi^2} = \left\{\beta^{3/2}x'' + \frac{\alpha^2}{\sqrt{\beta}}x + \left[\mathcal{K}\beta - \frac{1+\alpha^2}{\beta}\right]\sqrt{\beta}\,x\right\}\nu^2$$

$$= \left\{\beta^{3/2}\left[x'' + \mathcal{K}(s)\,x\right] - \frac{x}{\sqrt{\beta}}\right\}\nu^2$$

now: equation of motion:

$$x'' + \mathcal{K}(s)\,x = 0 \qquad \Big|\cdot \beta^{3/2}\nu^2$$

$$\Rightarrow \beta^{3/2}\left[x'' + \mathcal{K}(s)x\right]\nu^2 = 0 = \frac{d^2\eta}{d\phi^2} + \underbrace{\frac{x}{\sqrt{\beta}}}_{\eta}\nu^2$$

$$\Rightarrow \boxed{\frac{d^2\eta}{d\phi^2} + \nu^2\eta = 0}$$

transformed equ. of motion

solution: $\eta = \sqrt{2J}\,\sin\left(\nu(\phi + \phi_0)\right)$

step 2: Floquet's transformation of the inhomogeneous equation of motion

field errors, non-linear beam optics $\Rightarrow$ inh. equ. of motion

for horizontal motion: $\frac{q}{p} B_z = \frac{q}{p} B_{z,0} + \frac{q}{p} g x + \frac{q}{p} \Delta B(x,z,s)$

$$= \underbrace{\frac{1}{\rho(s)} + k(s) x}_{\text{linear beam optics}} + \frac{q}{p} \Delta B$$

$\Rightarrow$ equ. of motion: $x''(s) + K(s) x(s) = -\frac{q}{p} \Delta B$

$\Rightarrow$ Floquet's transformation:

$$\boxed{\frac{d^2\eta}{d\phi^2} + \nu^2 \eta = -\beta^{3/2} \nu^2 \frac{q}{p} \Delta B}$$

$\Rightarrow$ assume $z = 0$, expand ΔB in terms of multipoles:

$$\Delta B(x,s) = \Delta B_0 + \left.\frac{d\Delta B}{dx}\right|_{x=0} x + \frac{1}{2} \left.\frac{d^2 \Delta B}{dx^2}\right|_0 x^2 + \ldots$$

$\Rightarrow$ with $\frac{d}{d\eta} = \frac{d}{dx} \frac{dx}{d\eta} = \sqrt{\beta} \frac{d}{dx}$

$$\Rightarrow \Delta B(\eta) = \Delta B_0 + \left.\frac{d\Delta B}{d\eta}\right|_0 \eta + \frac{1}{2} \frac{d^2 \Delta B}{d\eta^2} \eta^2 + \ldots$$

$$\Rightarrow \frac{d^2\eta}{d\phi^2} + \nu^2 \eta = -\beta^{3/2} \nu^2 \frac{q}{p} \left\{ \underbrace{\Delta B_0}_{\text{dipole}} + \underbrace{\frac{d\Delta B}{d\eta} \eta}_{\text{quadrupole}} + \underbrace{\frac{1}{2} \frac{d^2 \Delta B}{d\eta^2} \eta^2}_{\text{sextupole}} + \ldots \right\}$$