Lecture 17

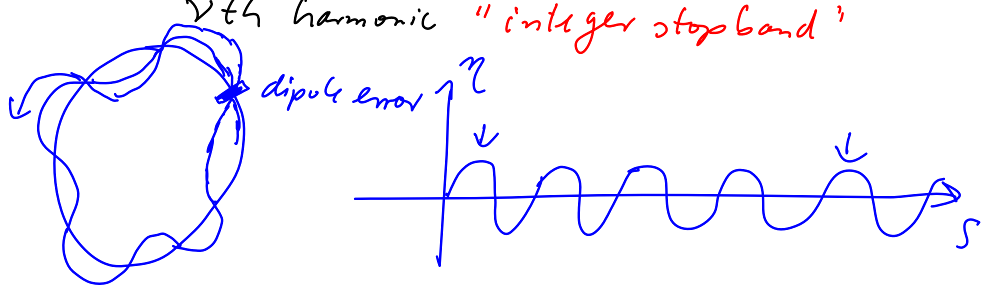
4. Beam optics in circular accelerators

4.12 Optical resonances

$$\frac{d^2\eta}{d\phi^2} + \nu^2\eta = -\beta^{3/2}\nu^2\frac{q}{p}\left\{\underbrace{\Delta B_0}_{\text{dipole}} + \underbrace{\frac{d\Delta B}{d\eta}\Big|_0 \eta}_{\text{quad}} + \underbrace{\frac{1}{2}\frac{d^2\Delta B}{d\eta^2}\Big|_0 \eta^2}_{\text{sextupole}} + \dots\right\}$$

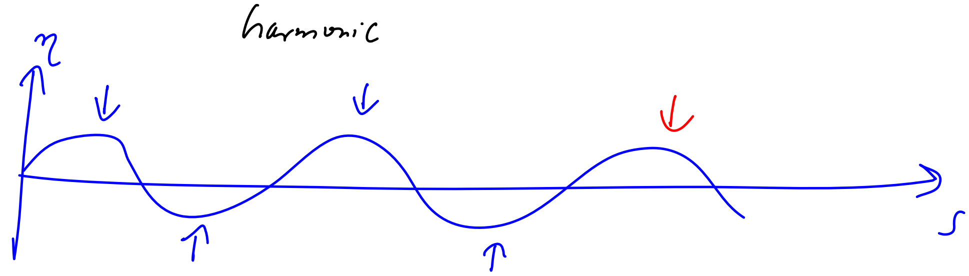
dipole term: resonance excitation for $\nu =$ integer if $\Delta B_0(s)$ has a non-zero term of the νth harmonic "integer stopband"

$\nu = 4$



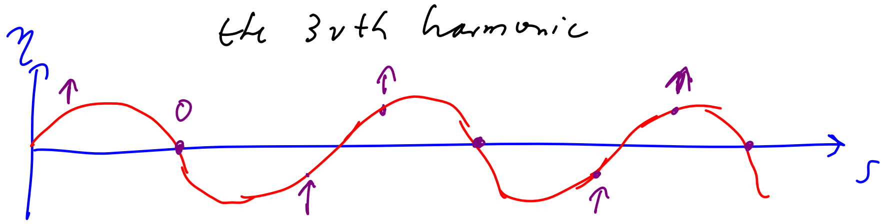
quadrupole term: resonance for $2\nu = \text{integer}$, if $\frac{d\Delta B}{d\eta}\eta$ has a non-zero term of the 2νth harmonic

$\nu = 1/2$



sextupole term: resonance for $3\nu = \text{integer}$, if $\frac{d^2\Delta B}{d\eta^2}\eta^2$ has a non-zero term of the 3νth harmonic

$\nu = \frac{1}{3}$



general: resonant excitation of beam if $\boxed{m\nu = p}$

$p, m = \text{integer}$

$\Rightarrow$ need to stay away from these tunes!

step 3: change of oscillation amplitude with field errors

solution of equation of motion : particle trajectory

$$u(s) = \sqrt{2J}\,\sqrt{\beta(s)}\,\cos\chi(s) \qquad \text{with } \chi = \psi(s) + \psi_0$$

$$\Rightarrow \quad u'(s) = -\frac{\sqrt{2J}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\chi(s) + \sin\chi(s)\right]$$

$\Rightarrow$ define $a = u(0) =$ amplitude of betatron oscillation at $s=0$

$$\beta_0 = \beta(0)$$

$$\Rightarrow \quad \sqrt{2J} = a/\sqrt{\beta_0}$$

$$\Rightarrow \quad u(s) = a\sqrt{\frac{\beta(s)}{\beta_0}}\cos\chi(s)$$

$$\Rightarrow \quad \beta(s)\,u'(s) = -a\sqrt{\frac{\beta(s)}{\beta_0}}\left[\alpha(s)\cos\chi(s) + \sin\chi(s)\right]$$

$$= \underbrace{-a\sqrt{\frac{\beta(s)}{\beta_0}}\cos\chi(s)}_{u(s)}\,\alpha(s) - a\sqrt{\frac{\beta(s)}{\beta_0}}\sin\chi(s)$$

$\Rightarrow$ replace $u'(s)$ by new variable:

$$w(s) \equiv \beta(s)\,u'(s) + \alpha(s)\,u(s) = -a\sqrt{\frac{\beta(s)}{\beta_0}}\sin\chi(s)$$

$\Rightarrow$ particle traces out circle in the u-w phase space plane at $s=0$ with (constant) radius = amplitude a

now: include field error $\Delta B(u,s)$ which acts over distance ds

$\Rightarrow$ induces change in angle (kick):

$$\Delta u'(\tilde{s}) = -\frac{q}{p}\,\Delta B(u,\tilde{s})\,d\tilde{s}$$

$\Rightarrow$ change in variable $w(s)$

$$\Delta w(\tilde{s}) = -\beta(\tilde{s})\,\frac{q}{p}\,\Delta B(u,\tilde{s})\,d\tilde{s}$$

- alternatively, in general form

$$\Delta u(\tilde{s}) = \left.\frac{du}{da}\right|_{\tilde{s}} \Delta a + \left.\frac{du}{d\chi}\right|_{\tilde{s}} \Delta\chi = 0 \qquad (1)$$

$$(2) \qquad \Delta w(\tilde{s}) = \left.\frac{dw}{da}\right|_{\tilde{s}} \Delta a + \left.\frac{dw}{da}\right|_{\tilde{s}} \Delta\chi = -\beta(\tilde{s})\frac{q}{p}\Delta B(u,\tilde{s})\,d\tilde{s}$$

from (1)

$$\Delta u = \sqrt{\frac{\beta(\tilde{s})}{\beta_0}}\left[\cos\chi(\tilde{s})\,\Delta a - a\sin\chi(\tilde{s})\,\Delta\chi\right] = 0$$

from (2)

$$\Delta w = -\sqrt{\frac{\beta(\tilde{s})}{\beta_0}}\left[\sin\chi(\tilde{s})\,\Delta a + a\cos\chi(\tilde{s})\,\Delta\chi\right]$$

$$= -\beta(\tilde{s})\,\frac{q}{p}\,\Delta B(u,\tilde{s})\,d\tilde{s}$$

$\Rightarrow$ solve for Δa, $\Delta\chi$

$$\Delta a = \sqrt{\beta_0\,\beta(\tilde{s})}\;\frac{q}{p}\,\Delta B(u,\tilde{s})\,\sin\chi(\tilde{s})\,d\tilde{s}$$

$$\Delta\chi = \frac{\sqrt{\beta_0\,\beta(\tilde{s})}}{a}\;\frac{q}{p}\,\Delta B(u,\tilde{s})\,\cos\chi(\tilde{s})\,d\tilde{s}$$

$\Rightarrow$ change of amplitude a per revolution:

with $x(s) = \Psi(s) + \Psi_0 = \nu\,\phi(s) + \Psi_0$

where $\phi(s) = \dfrac{\Psi(s)}{\nu} = \dfrac{1}{\nu}\displaystyle\int_0^s \frac{1}{\beta(s)}\,ds$

$$\boxed{\Delta a_{turn} = \frac{\Delta a_{n\,turns}}{n} = \frac{da}{dn} = \frac{\sqrt{\beta_0}\,q}{p}\oint \sqrt{\beta(\tilde{s})}\,\Delta B(u,\tilde{s})\cdot \sin\{\nu\,\phi(\tilde{s}) + \Psi_0\}\,d\tilde{s}}$$

$\Rightarrow$ only stable if $da/dn = 0$

$\Rightarrow$ if $da/dn \neq 0 \Rightarrow$ amplitude increases with each turn

$\Rightarrow$ resonance / instability

Example 1: Integer resonance $\nu = p =$ integer

assume dipole field error: $\Delta B(u,s) = \Delta B_0(s)$

$$\Rightarrow \frac{da}{dn} = \frac{\sqrt{\beta_0}\,q}{p}\oint \sqrt{\beta(\tilde{s})}\,\Delta B_0(\tilde{s})\,\sin\{\nu\,\phi(\tilde{s}) + \Psi_0\}\,d\tilde{s}$$

$$= \frac{\sqrt{\beta_0}\,q}{p}\oint \underbrace{\sqrt{\beta}\,\Delta B_0(\tilde{s})}\left[\sin\nu\phi\,\cos\Psi_0 + \cos\nu\phi\,\sin\Psi_0\right]d\tilde{s}$$

function $F(s) = \sqrt{\beta(s)}\,\Delta B_0(s)$ is periodic, with period = one revolution

$\Rightarrow$ can be written as a fourier series:

$$F(s) = F_0 + \sum_{p=1}^{\infty}\left[a_p \cos(p\,\phi(s)) + b \sin(p\,\phi(s))\right]$$

$\uparrow$ ϕ goes from 0 to 2π per turn

$\Rightarrow$ with $\int_0^{2\pi} \cos mx \sin nx \, dx = 0$

$$\frac{da}{dn} = \frac{\sqrt{\beta_0}\, q}{p}\left\{ \oint F_0 \left[\sin\nu\phi \cos\psi_0 + \cos\nu\phi \sin\psi_0 \right] d\tilde{s} \right.$$

$$\left. + \oint \sum_{p=1}^{\infty} \left[a_p \sin\psi_0 \cos p\phi \cos\nu\phi + b_p \cos\psi_0 \sin p\phi \sin\nu\phi \right] d\tilde{s} \right\}$$

since: $\int_0^{2\pi} \cos mx \cos nx \, dx \neq 0$ only if $m = n$

$\int_0^{2\pi} \sin mx \sin nx \, dx \neq 0$ only if $m = n$

$\Rightarrow$ only terms with $p = \nu$ contribute to $\frac{da}{dn}$

$\Rightarrow$ if tune of circular accelerator = integer:

$$\frac{da}{dn} = \frac{\sqrt{\beta_0}\, q}{p} \oint \left(a_\nu \sin\psi_0 \cos^2\nu\phi + b_\nu \cos\psi_0 \sin^2\nu\phi \right) d\tilde{s}$$

$\Rightarrow \frac{da}{dn} \neq$ if $a_\nu \neq 0$ or $b_\nu \neq 0$

$\Rightarrow$ resonance ("integer stopband") if $\nu = p$, with p = integer

$\Rightarrow$ strongest of all optical resonances

$\Rightarrow$ tune must be choosen sufficiently far from any integer value!

Example 2: Half-integer resonances : $2\nu = p$

consider quadrupole field error: $\Delta B(u,s) = g(s)\,u(s)$

use unperturbed trajectory

$$u(s) = a\sqrt{\frac{\beta(s)}{\beta_0}}\cos[\nu\phi(s)+\psi_0]$$

$\Rightarrow$

$$\frac{da}{dn} = \frac{aq}{p}\oint \beta(\tilde{s})\,g(\tilde{s})\cos[\nu\phi(\tilde{s})+\psi_0]\sin[\nu\phi(\tilde{s})+\psi_0]\,d\tilde{s}$$

$$= \frac{aq}{2p}\oint \underbrace{\beta(\tilde{s})\,g(\tilde{s})}\left[\sin 2\psi_0\cos \underline{\underline{2}}\nu\phi(\tilde{s}) + \cos 2\psi_0 \sin \underline{\underline{2}}\nu\phi(\tilde{s})\right]d\tilde{s}$$

$F = \beta(s)g(s)$ is again a periodic function with period = 1 turn; in general with pth harmonic $\neq 0$ $\Rightarrow$ $\cos(p\phi)$; $\sin(p\phi)$ terms

$$\Rightarrow \quad \frac{da}{dn} \neq 0 \quad , \text{if} \quad p = 2\nu$$

$\Rightarrow$ half integer resonances:

$$\boxed{2\nu = p \text{ with } p = \text{integer}}$$

Example 4: 1/3 integer resonances: $3\nu = p$

consider sextupole field: $\Delta B(u,s) = \frac{1}{2} g'(s)\, x^2(s)$

$\uparrow$ insert unperturbed trajectory

$$\Rightarrow \Delta B(u,s) = \frac{1}{2} g'(s)\, a^2 \frac{\beta(s)}{\beta_0} \cos^2[\nu\phi(s) + \psi_0]$$

$\Rightarrow$

$$\frac{da}{dn} = \frac{a^2 q}{2p\sqrt{\beta_0}} \oint \beta(s)^{3/2} g'(s) \cos^2[\nu\phi + \psi_0] \sin[\nu\phi + \psi_0]\, d\tilde{s}$$

use $\cos^2 x \sin x = \frac{1}{4}(\sin 3x + \sin x)$

$$\Rightarrow \frac{da}{dn} = \frac{a^2 q}{8p\sqrt{\beta_0}} \Big\{ \oint \beta^{3/2} g' [\sin\psi_0 \cos\nu\phi + \cos\psi_0 \sin\nu\phi]\, d\tilde{s} + \oint \beta^{3/2} g' [\sin 3\psi_0 \cos \underline{3\nu}\phi + \cos 3\psi_0 \sin \underline{3\nu}\phi]\, d\tilde{s} \Big\}$$

$\Rightarrow$ 1^{st} integral: integer resonances for $\nu = p$

2^{nd} integral: if periodic function $\beta^{3/2} g'$ has non-zero pth harmonic $\rightarrow$ terms $\cos p\phi$, $\sin p\phi$

$$\Rightarrow \frac{da}{dn} \neq 0 \quad \text{if} \quad 3\nu = p$$

$\Rightarrow$ 1/3 integer resonances

$$\boxed{3\nu = p \quad \text{, with } p = \text{integer}}$$

Tune diagram

higher order magnetic multipole $\rightarrow$ resonant conditions

dipole	$\longrightarrow$	$\nu = p$
quadrupole	$\longrightarrow$	$2\nu = p$
sextupole	$\longrightarrow$	$3\nu = p$
octupole	$\longrightarrow$	$4\nu = p$
$\boxed{2m\text{-pole}}$	$\longrightarrow$	$\boxed{m\nu = p}$

p = integer

rule of thumb: strength of resonance decreases strongly with the multipole order

· horizontal + vertical tune: ν_x, ν_z

multipoles couple planes! $\Rightarrow$ coupled resonances!

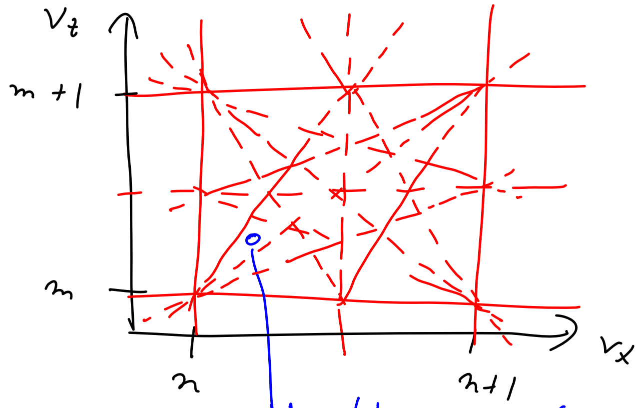
$\Rightarrow$ condition for optical resonances in both planes:

$$\boxed{m\nu_x + n\nu_z = p}$$

with m, n, p = integers, including 0, neg. values

$\Rightarrow$ sum $|m| + |n|$ = order of resonance

$\Rightarrow$ tune diagram: (up to 3^{rd} order)



working point (ν_x, ν_z)
sufficiently far away from resonances

Computer Lab next week

- Bring your Wille book
- Bring your laptop with matlab or matlab clone installed (see next slide)
- We will simulate the storage ring shown on page 97 in the textbook: