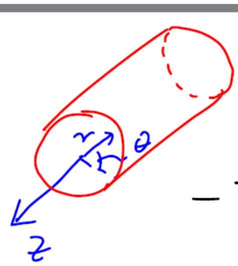
Lecture 19

5. RF Systems and Particle Acceleration

- **5.1 Waveguides**
  - **5.1.3 Cylindrical Waveguides**
- **5.2 Accelerating RF Cavities**
  - **5.2.1 Introduction**
  - **5.2.2 Traveling wave cavity: disk loaded waveguide**
  - **5.2.3 Standing wave cavities**
  - **5.2.4 Higher-Order-Modes**
  - **5.2.5 The pillbox cavity**
  - **5.2.6 SRF primer**

5.1.3 Circular Waveguide



use cylindrical coordinate system: r, θ, z

R = radius of waveguide

– TM modes: $B_z = 0$

$$\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \partial_t^2 E$$

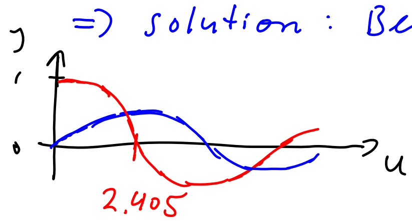
$$\Rightarrow \left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \partial_z^2 \right) E_z = \frac{1}{c^2} \partial_t^2 E_z$$

Try: $E_z = E_0(r)\, e^{in\theta}\, e^{ik_z z}\, e^{-i\omega t}$

with n = integer

$$\Rightarrow \left\{ \partial_r^2 + \frac{1}{r} \partial_r + \left(\left(\frac{\omega}{c}\right)^2 - \frac{n^2}{r^2} - k_z^2 \right) \right\} E_z = 0$$

$\Rightarrow$ solution: Bessel functions $J_n(u)$



Boundary condition: need $E_z = 0$ at wall

$\Rightarrow$ need zero of Bessel function at wall

$$\Rightarrow E_z = E_0 J_n(k_{nm} r) e^{in\theta} e^{ik_z z} e^{-i\omega t}$$

with $k_{nm} = \frac{z_{nm}}{R}$ ← m^{th} zero of the n^{th} Bessel function

$\Rightarrow TM_{nm}$ modes

- TE modes: $E_z = 0$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \partial_t^2 \vec{B}$$

at wall: need $\partial_r B_z = 0 \Rightarrow \ldots$

$$B_z = B_0 J_n(k_{nm} r) e^{in\theta} e^{ik_z z} e^{-i\omega t}$$

with $k_{nm} = \frac{S_{nm}}{R}$ ← m^{th} maximum of the n^{th} Bessel function

$\Rightarrow TE_{nm}$ modes

- Cut-off frequency:

$$\underline{\omega_c} = c k_c = \underline{c k_{nm}}$$

Fundamental Mode

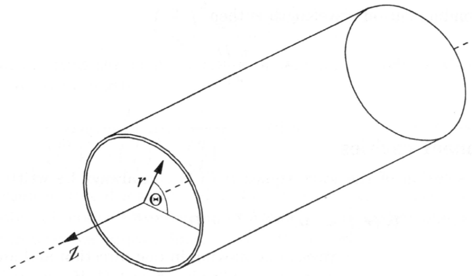
Mode for particle acceleration: TM_{01} $E_z(\vec{x}) = E_z J_0(\frac{r}{r_0})\cos(k_z z - \omega t)$

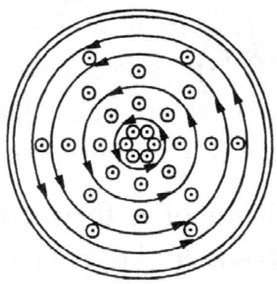
$$E_r(\vec{x}) = -E_z r_1 k_z J_0{}'(\tfrac{r}{r_1})\sin(k_z z - \omega t)$$

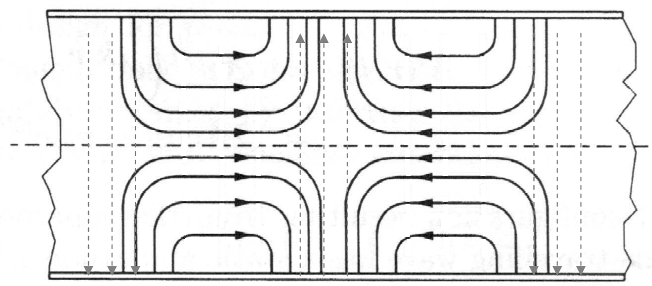
$$E_\varphi(\vec{x}) = 0$$

$$B_r(\vec{x}) = 0$$

$$B_\varphi(\vec{x}) = -E_z r_1 \tfrac{\omega}{c^2} J_0{}'(\tfrac{r}{r_1})\sin(k_z z - \omega t)$$





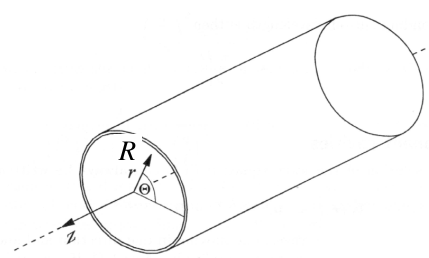


Cylindrical Wave TE Modes

$$E_z(\vec{x}) = 0\,, \quad B_z(\vec{x}) = B_0 J_n(\tfrac{S_{nm}}{R} r)e^{in\varphi}$$

$$\vec{E}_\perp = \frac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z \vec{\nabla}_\perp E_z + \omega \vec{\nabla}_\perp \times \vec{B}_z)$$

$$\vec{B}_\perp = \frac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z \vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2} \vec{\nabla}_\perp \times E_z)$$



$$E_r = i\omega\left(\tfrac{R}{S_{nm}}\right)^2 \tfrac{1}{r}\partial_\varphi B_z = -B_0 n\omega R \tfrac{1}{S_{nm}^2}\tfrac{R}{r} J_n(\tfrac{S_{nm}}{R} r)\cos(n\varphi + k_z z - \omega t)$$

$$E_\varphi = -i\omega\left(\tfrac{R}{S_{nm}}\right)^2 \partial_r B_z = B_0 \omega R \tfrac{1}{S_{nm}} J'_n(\tfrac{S_{nm}}{R} r)\sin(n\varphi + k_z z - \omega t)$$

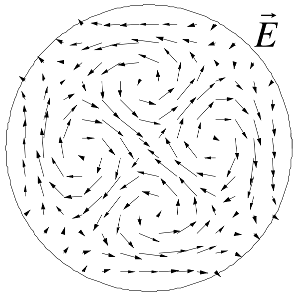
$$E_z = 0$$

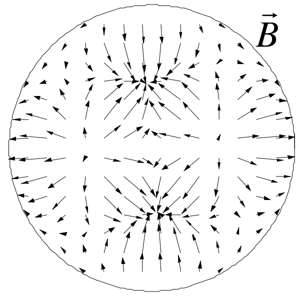
$$B_r = ik_z\left(\tfrac{R}{S_{nm}}\right)^2 \partial_r B_z = -B_0 k_z R \tfrac{1}{S_{nm}} J'_n(\tfrac{S_{nm}}{R} r)\sin(n\varphi + k_z z - \omega t)$$

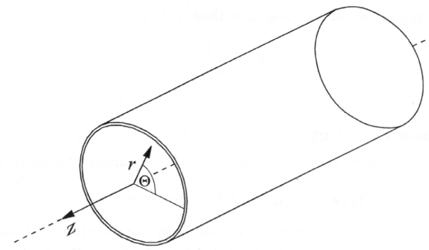
$$B_\varphi = ik_z\left(\tfrac{R}{S_{nm}}\right)^2 \tfrac{1}{r}\partial_\varphi B_z = -B_0 n k_z R \tfrac{1}{S_{nm}^2}\tfrac{R}{r} J_n(\tfrac{S_{nm}}{R} r)\cos(n\varphi + k_z z - \omega t)$$

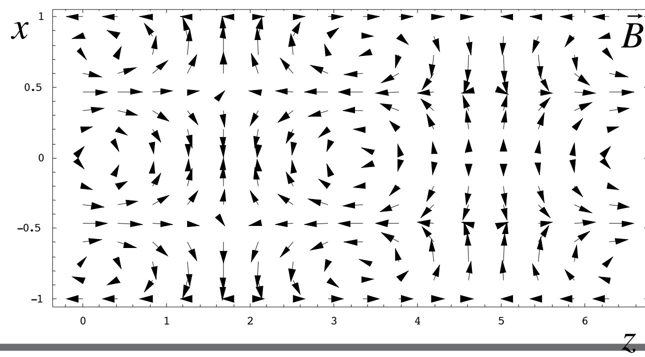
$$B_z = B_0 J_n(\tfrac{S_{nm}}{R} r)\cos(n\varphi + k_z z - \omega t)$$

Example: Cylindrical TE_{22} Mode







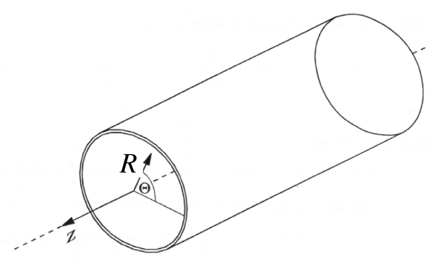


Cylindrical Wave TM Modes

$$E_z(\vec{x}) = E_0 J_n(\tfrac{Z_{nm}}{R} r) e^{in\varphi} \;, \quad B_z(\vec{x}) = 0$$

$$\vec{E}_\perp = \frac{i}{\frac{\omega^2}{c^2} - k_z^2} (k_z \vec{\nabla}_\perp E_z + \omega \vec{\nabla}_\perp \times \vec{B}_z)$$

$$\vec{B}_\perp = \frac{i}{\frac{\omega^2}{c^2} - k_z^2} (k_z \vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2} \vec{\nabla}_\perp \times \vec{E}_z)$$



$$E_r = i \tfrac{R k_z}{Z_{nm}} \partial_r E_z = -E_0 k_z J'_n(\tfrac{Z_{nm}}{R} r) \sin(n\varphi + k_z z - \omega t)$$

$$E_\varphi = i \tfrac{R k_z}{Z_{nm}} \tfrac{1}{r} \partial_\varphi E_z = -E_0 n k_z \tfrac{1}{Z_{nm}} \tfrac{R}{r} J_n(\tfrac{Z_{nm}}{R} r) \cos(n\varphi + k_z z - \omega t)$$

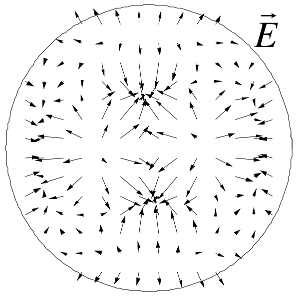
$$E_z = E_0 J_n(\tfrac{Z_{nm}}{R} r) \cos(n\varphi + k_z z - \omega t)$$

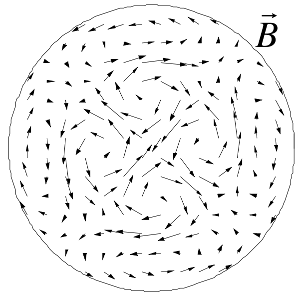
$$B_r = -i \tfrac{R\omega}{Z_{nm}} \tfrac{1}{r} \partial_\varphi E_z = E_0 n \tfrac{\omega}{c^2} \tfrac{1}{Z_{nm}} \tfrac{R}{r} J_n(\tfrac{Z_{nm}}{R} r) \cos(n\varphi + k_z z - \omega t)$$

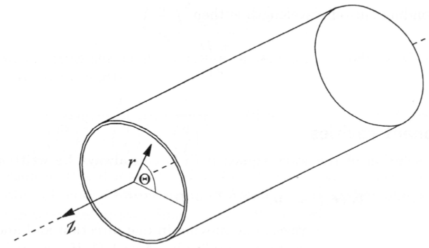
$$B_\varphi = i \tfrac{R\omega}{Z_{nm}} \partial_r E_z = -E_0 \tfrac{\omega}{c^2} J'_n(\tfrac{Z_{nm}}{R} r) \sin(n\varphi + k_z z - \omega t)$$

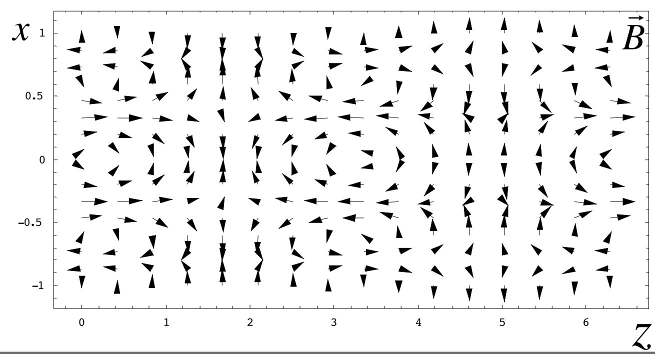
$$B_z = 0$$

Example: Cylindrical TM_{22} Mode

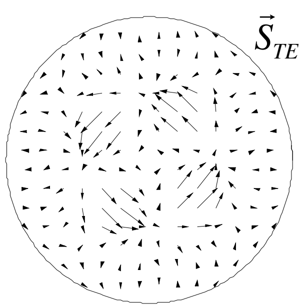


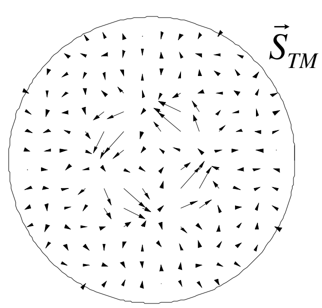


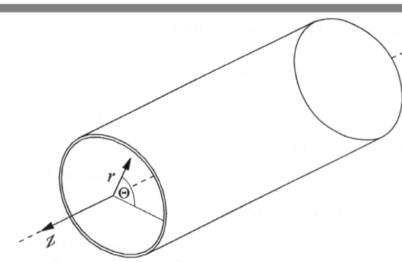


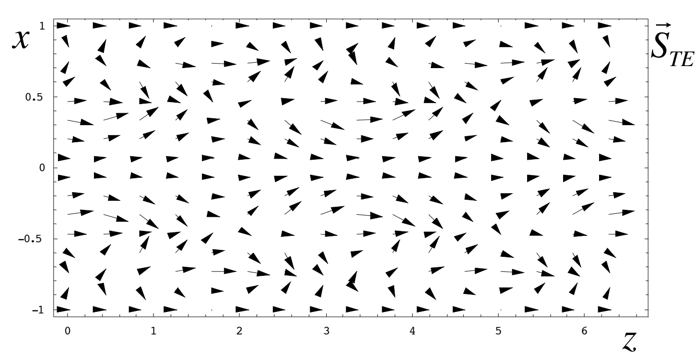


Pointing Vector

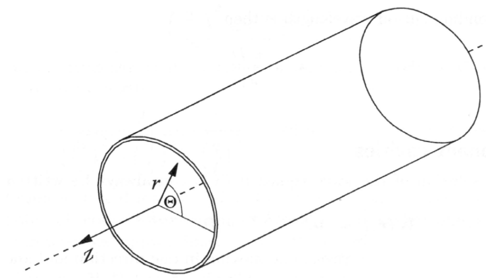


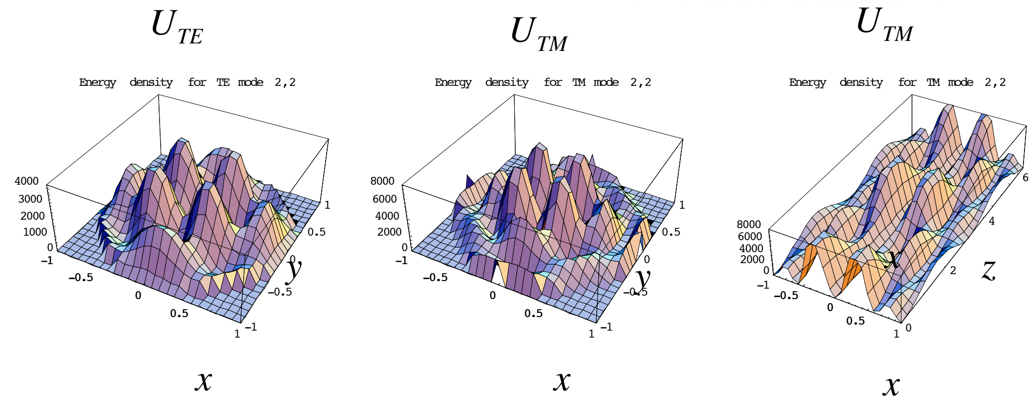




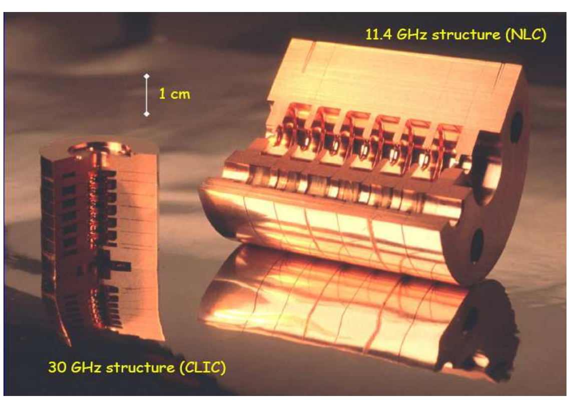


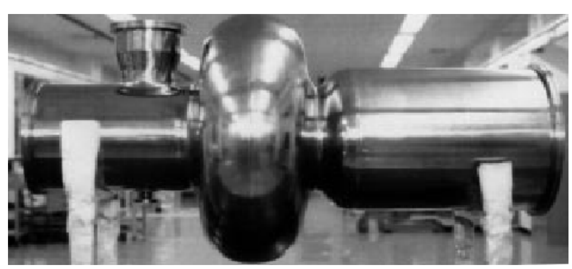
Energy Density





5.2 Accelerating RF Cavities



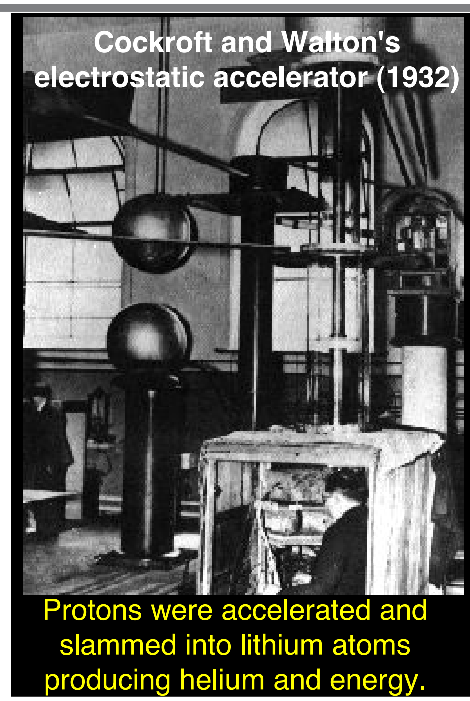


DC Accelerators:

- Use high DC voltage to accelerate particles

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

- No work done by magnetic fields



DC Accelerators: Limitations

1) DC ($\frac{\partial}{\partial t} \equiv 0$): $\nabla \times \vec{E} = 0$ which is solved by $\vec{E} = -\nabla\Phi$

Limit: If you want to gain 1 MeV, you need a potential of 1 MV!

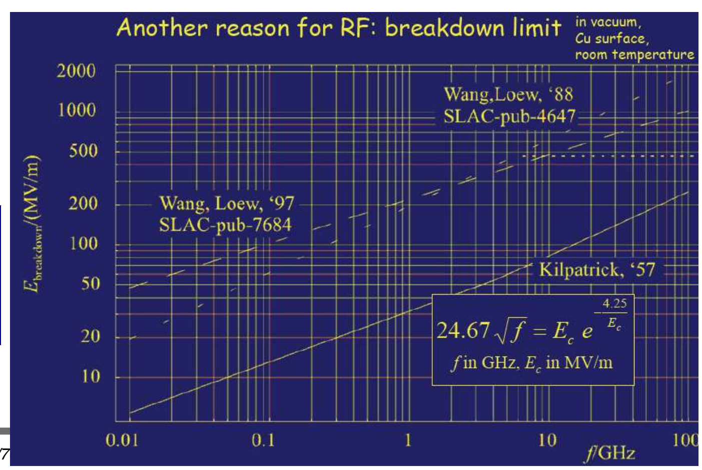
2) Circular machine: DC acceleration impossible since $\oint \vec{E} \cdot d\vec{s} = 0$

$\Rightarrow$ Use time-varying fields!

Maxwell's equation in vacuum (contd.)

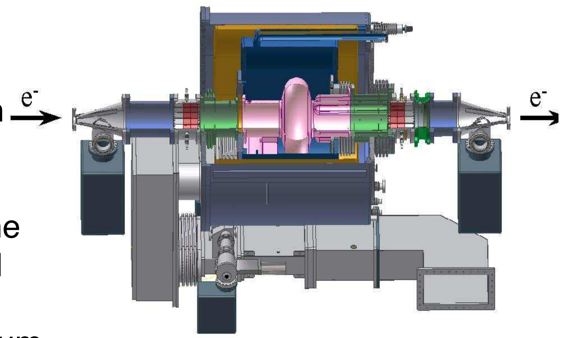
$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \qquad \nabla \cdot \vec{E} = 0$$



RF Cavities

- The main purpose of using RF cavities in accelerators is to provide energy gain to charged-particle beams
- The highest achievable gradient, however, is not always optimal for an accelerator. There are other factors (both machine-dependent and technology-dependent) that determine operating gradient of RF cavities and influence the cavity design, such as accelerator cost optimization, maximum power through an input coupler, necessity to extract HOM power, etc.
- In many cases requirements are competing.



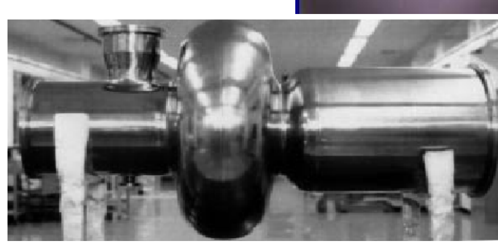
Taiwan Light Source cryomodule

CW High-Current Storage Rings (colliders and light sources)

- NC or SC
- Relatively low gradient
  (1…9 MV/m)
- Strong HOM damping
  (Q ~ 10^2)
- High average RF power
  (hundreds of kW)

CESR cavities

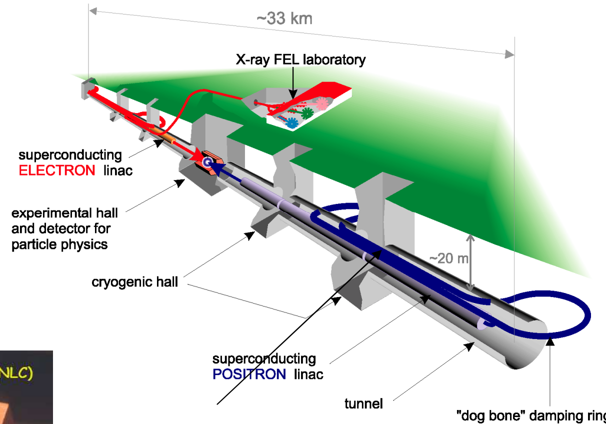
PEP II Cavity

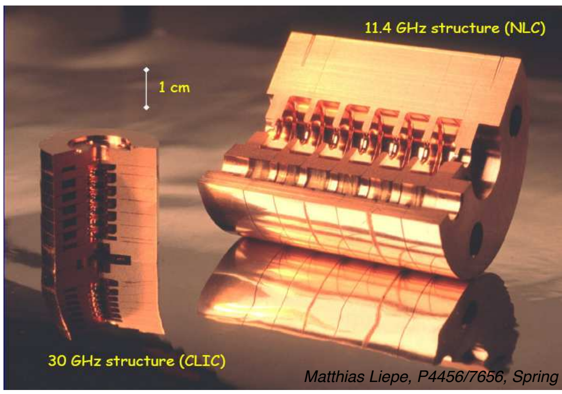


KEK cavity

Pulsed Linacs (ILC, XFEL, …)

- NC or SC
- High gradients
- Moderate HOM damping reqs.
- High peak RF power





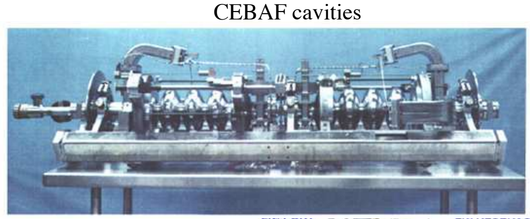
ILC: 21,000 cavities!

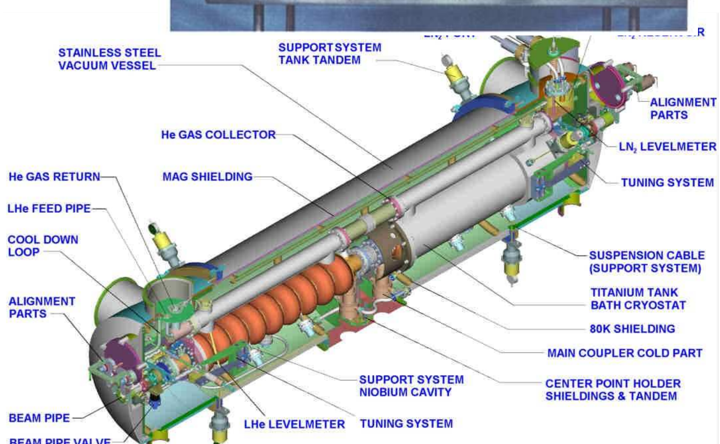
ILC / XFEL cavities

CW low-current linacs (CEBAF, ELBE)

- SRF cavities
- Moderate to low gradient (8…20 MV/m)
- Relaxed HOM damping requirements
- Low average RF power (5…13 kW)

CEBAF cavities

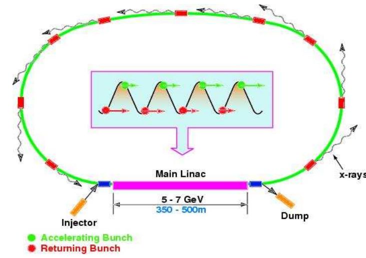


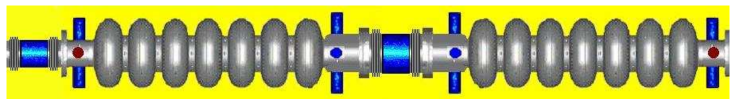


ELBE cryomodule

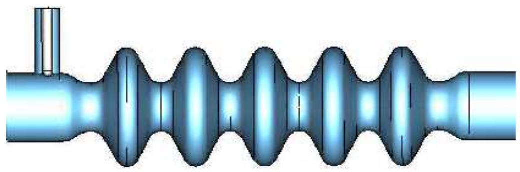
CW High-Current ERLs

- SRF cavities
- Moderate gradient (15…20 MV/m)
- Strong HOM damping ($Q = 10^2 \dots 10^4$)
- Low average RF power (few kW)





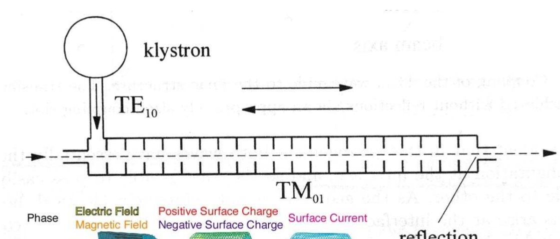
Cornell ERL cavities

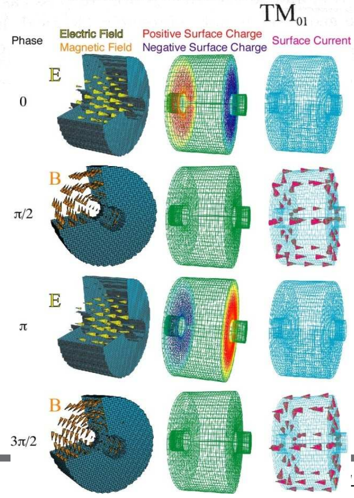


BNL ERL cavity

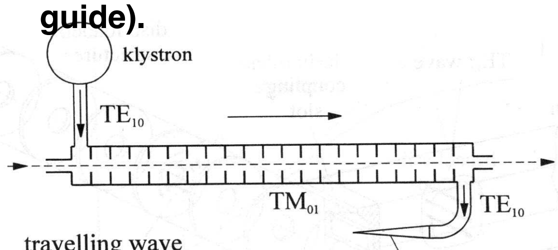
Two basic types of RF cavities

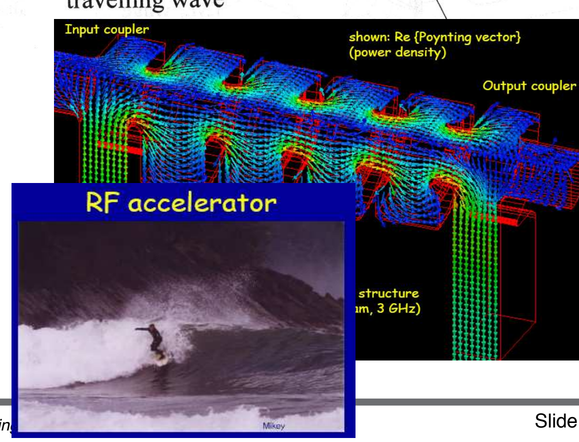
Standing wave cavity.





Traveling wave cavity (wave guide).



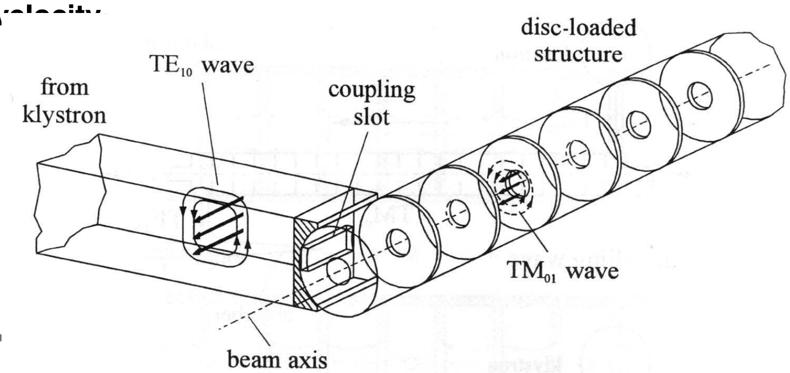


5.2.2 Traveling wave cavities

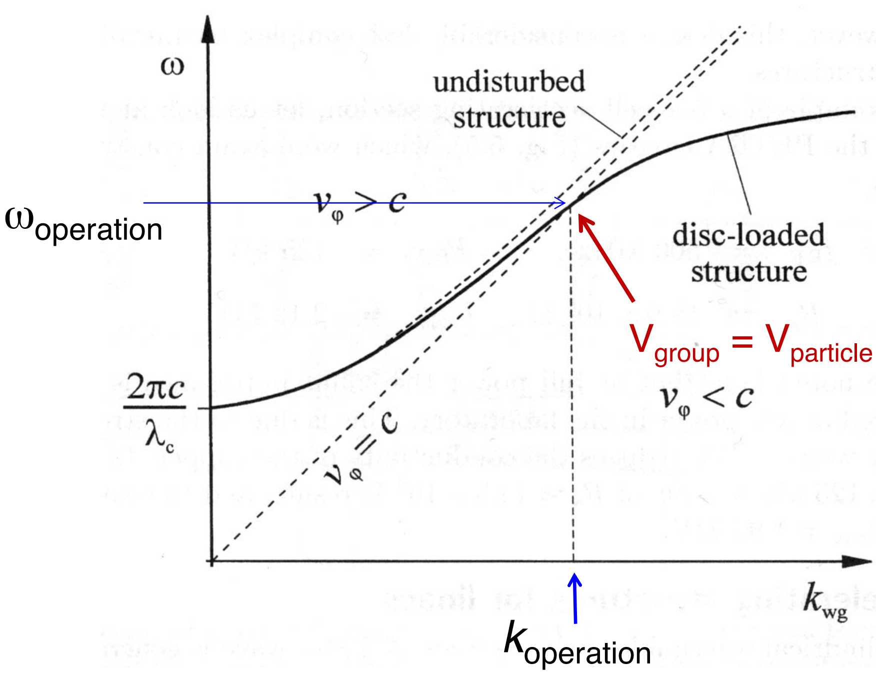
- **Cylindrical Waveguide: TM01 has longitudinal electric field and could in principle be used for particle acceleration**
- **But: phase velocity of wave > c > speed of particle**
  -> no average energy transfer to beam !

· phase velocity:
$v_{ph} = \frac{\omega}{k_z} = c\sqrt{1+\left(\frac{k_c}{k_z}\right)^2} > c$

- **Solution: Disc Loaded Waveguide**
  - **Iris shaped plates at constant separation in waveguide lower phase velocity**
  - **Iris size is chosen to make the phase velocity equal the particle velocity**



Disc Loaded Waveguide: Dispersion

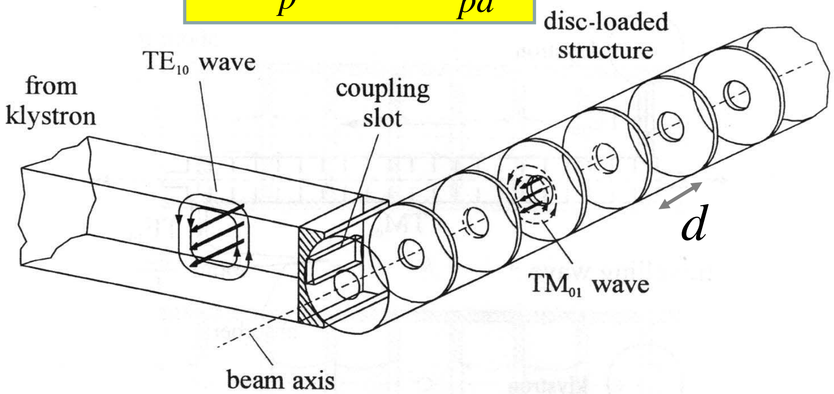


Disc Loaded Waveguide

- Irises form periodic structure in waveguide
  - -> Irises reflect part of wave
  - -> Interference
  - -> For loss free propagation: need disk spacing *d*

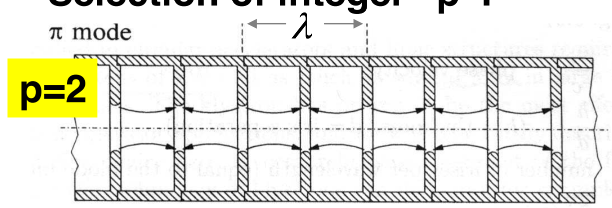
$$d = \frac{\lambda_z}{p} \Rightarrow k_z = \frac{2\pi}{pd}$$

p=integer

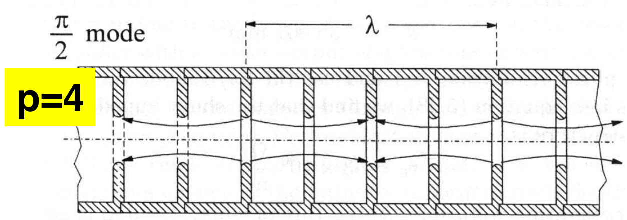


Disc Loaded Waveguide: $k_z/d=2\pi/p$ Modes

Selection of integer "p":

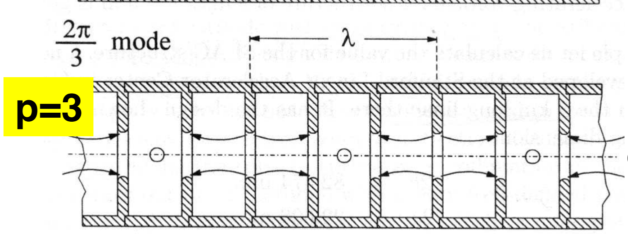


Long initial settling or filling time, not good for pulsed operation with very short pulses.



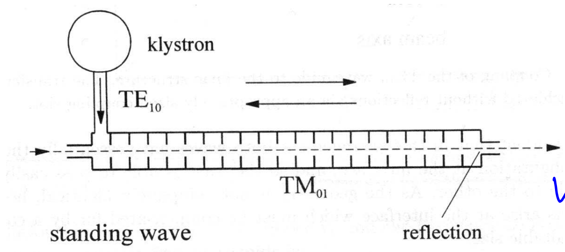
Small shunt impedance per length (shunt impedance determines how much acceleration a particle can get for a given power dissipation in a cavity).

$$R_{sh} = \frac{V_c^2}{P_c}$$



Common compromise.

5.2.3 Standing wave cavities



superposition of two waves, moving opposite direction

$\Rightarrow$ standing wave

$$W(\vec{r},t) = A\, e^{i\omega t}\left(e^{i\vec{k}\cdot\vec{r}} + e^{-i\vec{k}\cdot\vec{r}}\right) = 2A\cos(\vec{k}\cdot\vec{r})\, e^{i\omega t}$$

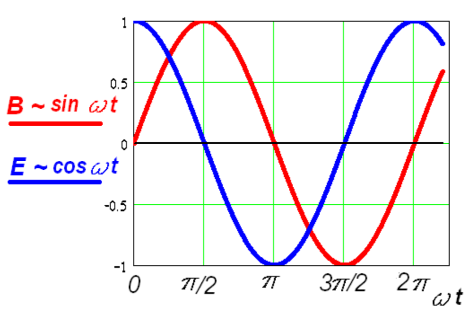
$\Rightarrow$ $W = 0$ at positions $\vec{k}\cdot\vec{r} = (l + \frac{1}{2})\pi$

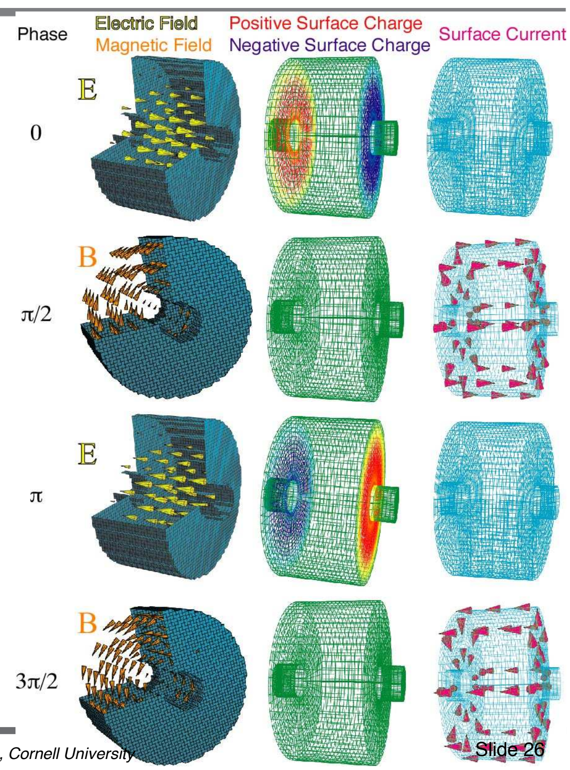
$\Rightarrow$ metal walls can be placed at certain positions without changing the field pattern

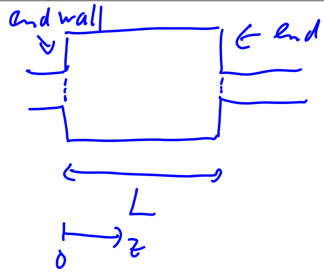
$\Rightarrow$ Resonant cavities!

Standing Wave RF Cavities

- Time dependent electromagnetic field inside metal box
- Energy oscillates between electric and magnetic field!







TE modes: need $B_z = 0$ at end walls

$\Rightarrow$ standing waves:

$$B_z \propto \sin(k_z z)\sin(\omega t) \Rightarrow TE_{nml} \text{ modes}$$

with $k_z = \frac{l\pi}{L}$, $l = 1, 2, 3, \ldots$

TM-modes: need $E_x, E_y = 0$ at walls

$\Rightarrow$ standing waves:

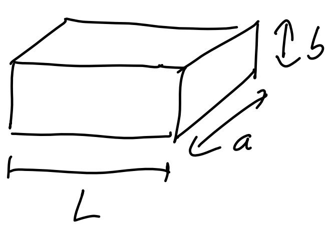
$$E_z \propto \cos(k_z z)\cos(\omega t) \Rightarrow TM_{nml} \text{ modes}$$

with $k_z = \frac{l\pi}{L}$, $l = 0, 1, 2, 3, \ldots$

$\Rightarrow$ set of resonance frequencies:

$$k^2 = \left(\frac{\omega}{c}\right)^2 = k_{c,nm}^2 + k_z^2 \Rightarrow \omega_{nml} = c\sqrt{k_{c,nm}^2 + \left(\frac{l\pi}{L}\right)^2}$$

Example: Rectangular waveguide → cavity



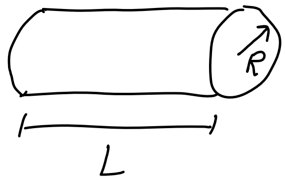
$$\omega_{nml} = c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{L}\right)^2}$$

$\Rightarrow$ fundamental accelerating mode: TM_{110}

$$\omega_{110} = c\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$

$\Rightarrow$ for $a = b = 21.2$ cm $\Rightarrow f_{110} = 1$ GHz

Example: Pill box cavity / cylindrical cavity



$$\omega_{nml} = c\sqrt{K_{nm}^2 + \left(\frac{l\pi}{L}\right)^2}$$

$K_{nm}R$: m^{th} zero of the n^{th} Bessel function for TM modes

$K_{nm}R$: m^{th} extremum of the n^{th} Bessel function for TE modes

⇒ fundamental accelerating mode: TM_{010}

$$\omega_{010} = c\,\frac{2.405}{R}$$

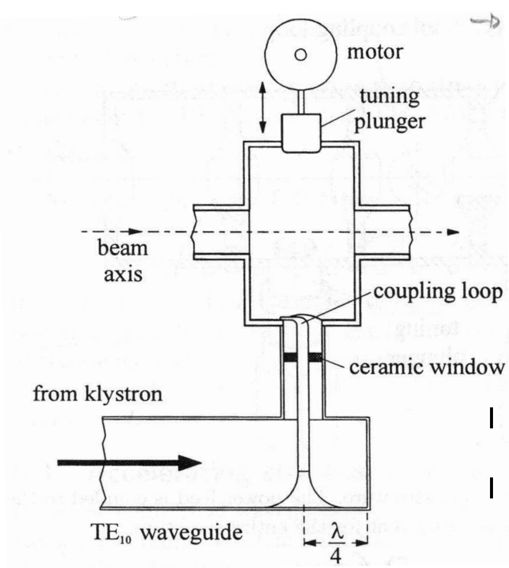
Example: $R = 11.45\,cm \Rightarrow f_{011} = 1\,GHz$

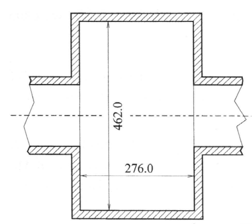
~ 3.95 GHz is the lowest frequency

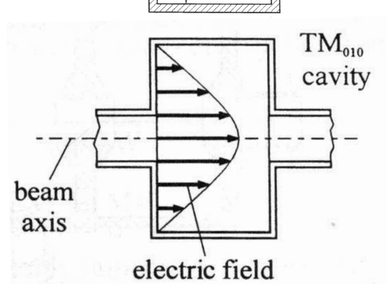
Cavity Operation

500MHz Cavity of DORIS:

$$r = 23.1cm \quad \Rightarrow \quad f_{010}^{(M)} = 0.4967\text{GHz}$$





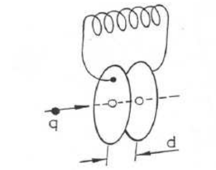


- The frequency is increased and tuned by a tuning plunger.
- An inductive coupling loop excites the magnetic field at the equator of the cavity.

Cavity resonator

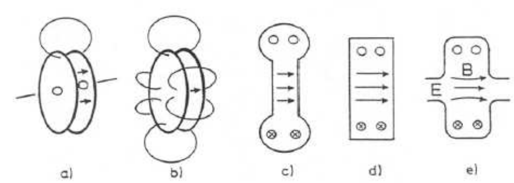
An LC circuit, the simplest form of RF resonator, as an accelerating device.

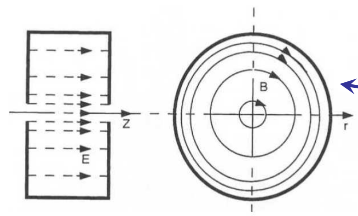
$$\omega = \frac{1}{\sqrt{LC}}$$



Metamorphosis of the LC circuit into an accelerating cavity:

1. Increase resonant frequency by lowering L, eventually have a solid wall.
2. Further frequency increase by lowering C → arriving at cylindrical, or "pillbox" cavity geometry, which can be solved analytically.
3. Add beam tubes to let particle pass through.





- Magnetic field is concentrated at the cylindrical wall, responsible for RF losses.
- Electric field is concentrated near axis, responsible for acceleration.

Cavity modes: TM010 used for acceleration

- Fields in the cavity are solutions of the equation

$$\left(\nabla^2 - \frac{1}{c}\frac{\partial^2}{\partial t}\right)\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} = 0$$

- Subject to the boundary conditions

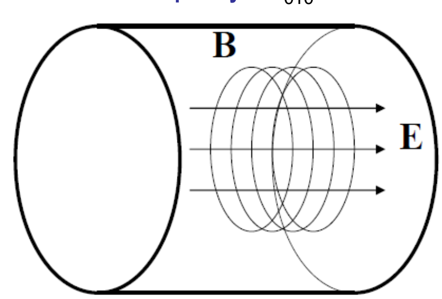
$$\hat{n} \times \mathbf{E} = 0, \quad \hat{n} \cdot \mathbf{H} = 0$$

- The infinite number of solutions (eigenmodes) belong to two families of modes with different field structure and eigenfrequencies: TE modes have only transverse electric fields, TM modes have only transverse magnetic fields.
- One needs longitudinal electric field for acceleration, hence the lowest frequency TM_{010} mode is used.
- For the pillbox cavity w/o beam tubes

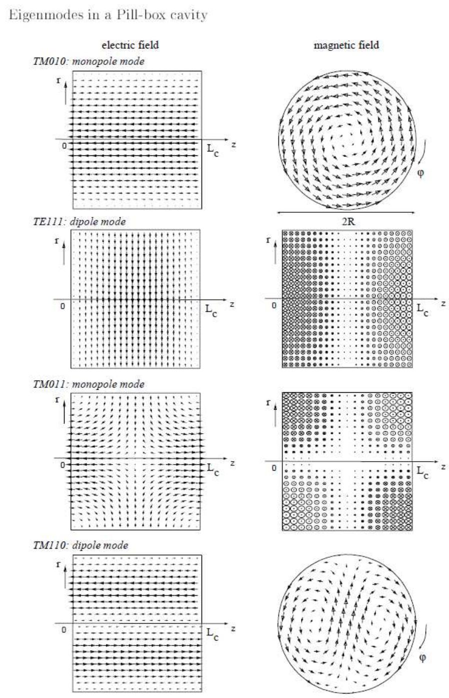
$$E_z = E_0 J_0\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

$$H_\varphi = -i\frac{E_0}{\eta} J_1\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

$$\omega_{010} = \frac{2.405c}{R}, \quad \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

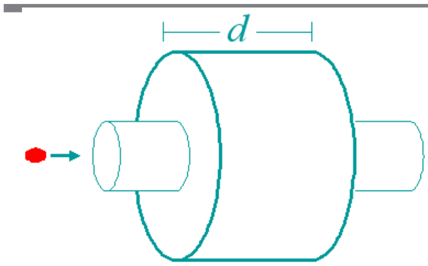


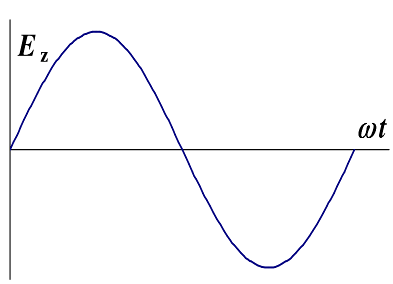
Higher Frequency (Order) Standing Wave Modes



- The modes are classified as TM_{mnp} (TE_{mnp}), where integer indicies *m, n,* and *p* correspond to the number of variations E_z (H_z) has in φ, *r,* and *z* directions respectively.
- While TM_{010} mode is used for acceleration and usually is the lowest frequency mode, all other modes are "parasitic" as they may cause various unwanted effects. Those modes are referred to as Higher-Order Modes (HOMs).

Accelerating voltage & transit time





- Assuming charged particles moving along the cavity axis, one can calculate accelerating voltage as

$$V_c = \left| \int_{-\infty}^{\infty} E_z(\rho = 0, z) e^{i\omega_0 z/\beta c} dz \right|$$

For the pillbox cavity one can integrate this analytically:

$$V_c = E_0 \left| \int_0^d e^{i\omega_0 z/\beta c} dz \right| = E_0 d \frac{\sin\left(\frac{\omega_0 d}{2\beta c}\right)}{\frac{\omega_0 d}{2\beta c}} = E_0 d \cdot T$$

where *T* is the transit time factor.

- To get maximum acceleration:

$$T_{transit} = t_{exit} - t_{enter} = \frac{T_0}{2} \Rightarrow d = \beta\lambda/2 \Rightarrow \quad V_c = \frac{2}{\pi} E_0 d$$

Thus for the pillbox cavity $T = 2/\pi$.

- The accelerating field E_{acc} is defined as $E_{acc} = V_c/d$. Unfortunately the cavity length is not easy to specify for shapes other than pillbox so usually it is assumed to be $d = \beta\lambda/2$. This works OK for multicell cavities, but poorly for single-cell ones.

Dissipated Power, Stored energy

- Surface currents ($\propto H$) result in dissipation proportional to the surface resistance (R_s):

$$\frac{dP_c}{ds} = \frac{1}{2} R_s |\mathbf{H}|^2$$

- Dissipation in the cavity wall given by surface integral:

$$P_c = \frac{1}{2} R_s \int_S |\mathbf{H}|^2 \, ds$$

- Energy density in electromagnetic field:

$$u = \frac{1}{2}\left(\varepsilon \cdot \mathbf{E}^2 + \mu \cdot \mathbf{H}^2\right)$$

- Because of the sinusoidal time dependence and 90° phase shift, the energy oscillates back and forth between the electric and magnetic field. The stored energy in a cavity is given by

$$U = \frac{1}{2}\mu_0 \int_V |\mathbf{H}|^2 dv = \frac{1}{2}\varepsilon_0 \int_V |\mathbf{E}|^2 dv$$

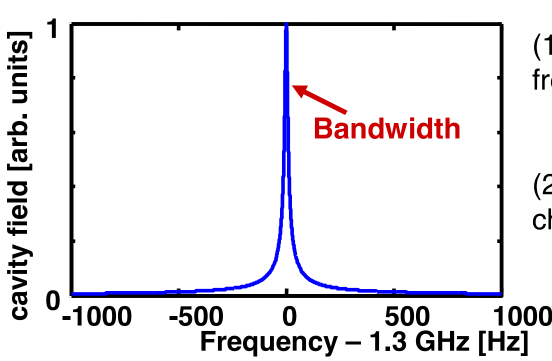
Quality factor

- **An important figure of merit is the quality factor, which for any resonant system is**

$$Q_0 \equiv \frac{\omega_0 \cdot (\text{stored energy})}{\text{average power loss}} = \frac{\omega_0 U}{P_c} = 2\pi \frac{1}{T_0}\frac{U}{P_c} = \omega_0 \tau_0 = \frac{\omega_0}{\Delta\omega_0}$$

$$Q_0 = \frac{\omega_0 \mu_0 \int_V |\mathbf{H}|^2 dv}{R_s \int_S |\mathbf{H}|^2 ds}$$

roughly 2π **times the number of RF cycles it takes to dissipate the energy stored in the cavity. It is determined by both the material properties and cavity geometry and** $\sim 10^4$ **for NC cavities and** $\sim 10^{10}$ **for SC cavities at 2 K.**



(1) The RF system has a resonant frequency ω_0

(2) The resonance curve has a characteristic width $\Delta\omega = \frac{\omega_0}{2Q}$

Geometry factor

- One can see that the ration of two integrals in the last equation determined only by cavity geometry. Thus we can re-write it as

$$Q_0 = \frac{G}{R_s}$$

with the parameter G known as the geometry factor or geometry constant

$$G = \frac{\omega_0 \mu_0 \int_V |\mathbf{H}|^2 dv}{\int_S |\mathbf{H}|^2 ds}$$

- The geometry factor depends only on the cavity shape and electromagnetic mode, but not its size. Hence it is very useful for comparing different cavity shapes. G = 257 Ohm for the pillbox cavity.

Plug in some numbers:

Copper: f = 1.5 GHz, σ = 5.8 x 10^7 A/Vm, μ_0 = 1.26 x 10^{-6} Vs/Am

→ δ = 1.7 µm, R_s = 10 mΩ
→ Q_0 = G/Rs = 25700

Shunt impedance and R/Q

- The shunt impedance determines how much acceleration a particle can get for a given power dissipation in a cavity

$$R_{sh} = \frac{V_c^2}{P_c}$$

It characterized the cavity losses. Units are Ohms. Often the shunt impedance is defined as in circuit theory

$$R_{sh} = \frac{V_c^2}{2P_c}$$

and, to add to the confusion, a common definition in linacs is

$$r_{sh} = \frac{E_{acc}^2}{P_c'}$$

where P'_c is the power dissipation per unit length and the shunt impedance is in Ohms per meter.

- A related quantity is the ratio of the shunt impedance to the quality factor, which is independent of the surface resistivity and the cavity size:

$$\frac{R_{sh}}{Q_0} = \frac{V_c^2}{\omega_0 U (2)}$$ ← for "circuit definition"

- This parameter is frequently used as a figure of merit and useful in determining the level of mode excitation by bunches of charged particles passing through the cavity. R/Q = 196 Ohm for the pillbox cavity. Sometimes it is called geometric shunt impedance.

Dissipated power

- The power loss in the cavity walls is

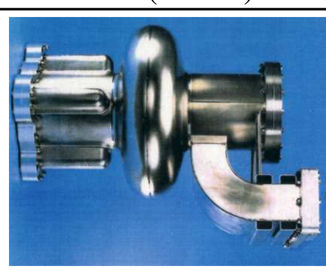
$$P_c = \frac{V_c^2}{R_{sh}} = \frac{V_c^2}{Q_0 \cdot (R_{sh}/Q_0)} = \frac{V_c^2}{(R_s \cdot Q_0)(R_{sh}/Q_0)/R_s} = \frac{V_c^2 \cdot R_s}{G \cdot (R_{sh}/Q_0)} \quad (2)$$

for "circuit def."

- To minimize the losses one needs to maximize the denominator. By modifying the formula, one can make the denominator material-independent: *G·R/Q* – this new parameter can be used during cavity shape optimization.

Pillbox vs. "real life" cavity

Quantity	Cornell SC 500 MHz	Pillbox
G	270 Ω	257 Ω
R_a/Q_0	88 Ω /cell	196 Ω/cell
E_{pk}/E_{acc}	2.5	1.6
H_{pk}/E_{acc}	52 Oe/(MV/m)	30.5 Oe/(MV/m)



- In a high-current storage ring, it is necessary to damp Higher-Order Modes (HOMs) to avoid beam instabilities.
- The beam pipes are made large to allow HOMs propagation toward microwave absorbers
- This enhances H_{pk} and E_{pk} and reduces *R/Q*.