Lecture 23

5. RF Systems and Particle Acceleration

5.4 Phase focusing and longitudinal (synchrotron) beam oscillation

Longitudinal motion: Synchrotron oscillations

$$\boxed{\frac{E_0 \beta_0^2}{\omega_0^2 h \eta} \frac{d^2\phi}{dt^2} = \frac{q\hat{V}}{2\pi}\left(\sin\phi - \sin\phi_0\right)} \quad (3)$$

diff.-eqn. for synchrotron oscillations $\phi(t)$ in adiabatic approximation: assume that $E_0, \omega_0, \beta_0, \eta$ are changing only slowly, i.e. are $\approx$ const within time scale T_0

$\Rightarrow$ this gives for small $\Delta\phi$:

$$\frac{d^2(\Delta\phi)}{dt} + \Omega^2 \Delta\phi = 0 \quad \text{with } \Omega^2 = -\eta \cos\phi_0 \frac{q\hat{V}}{2\pi} \frac{\omega_0^2 h}{E_0 \beta_0^2}$$

$\Rightarrow$ for $\Omega^2 > 0$: simple harmonic oscillations with frequency Ω: <u>synchrotron frequency</u>

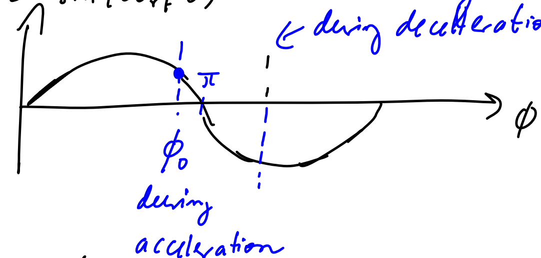
- $E > E_{tr}$ case: $\eta = \left(\frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2} \right) > 0$

$\Rightarrow$ need $\cos\phi_0 < 0$

$\Rightarrow$ need $\frac{\pi}{2} < \phi_0 < \pi$ for stability during acceleration

$V(t) = \hat{V} \sin(\omega_{rf} t)$

$\leftarrow$ during deceleration: $\pi < \phi_0 < \frac{3}{2}\pi$

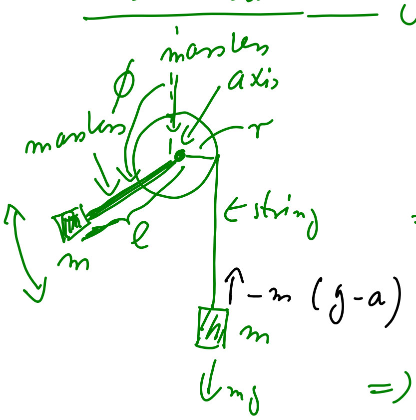


Note: during transition from $E < E_{tr}$ to $E > E_{tr}$ rf phase ϕ_0 must jump from $0 < \phi_0 < \frac{\pi}{2}$ to $\frac{\pi}{2} < \phi < \pi$

$\sim$ technically possible, since $\eta \to 0$ for $E \to E_{tr}$

$\Rightarrow$ $\Delta T \to 0$ and $\Omega^2 \propto \eta \to 0$, i.e. oscillations very slow near E_{tr}.

- Mechanical analogon: balanced pendulum



$$\frac{dL}{dt} = m l^2 \ddot{\phi} = \sum \tau$$

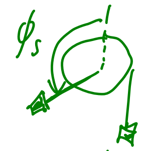
$$\Rightarrow m l^2 \ddot{\phi} = m g l \sin\phi - m g r + m a r$$

$$a = -\ddot{\phi} r$$

$$\Rightarrow \underbrace{(m l^2 + m r^2)}_{I} \ddot{\phi} = m g l \sin\phi - m g r$$

$\Rightarrow$ equilibrium points:

$$\ddot{\phi} = 0 \Rightarrow l \sin\phi_{equ.} = r \Rightarrow \sin\phi_{equ.} = r/l$$

- stable:  example: $\ell = 2r \Rightarrow \phi_s = 150°$

- unstable:  example: $\ell = 2r \Rightarrow \phi_u = 30°$

note: $\boxed{\phi_u = \pi - \phi_s}$

$\Rightarrow$ with $r = \ell \sin\phi_s$

$$\boxed{I\ddot{\phi} = mg\ell(\sin\phi - \sin\phi_s)}$$

$\rightsquigarrow$ analogon to diff. equation (3) for synchrotron oscillations about ϕ_0

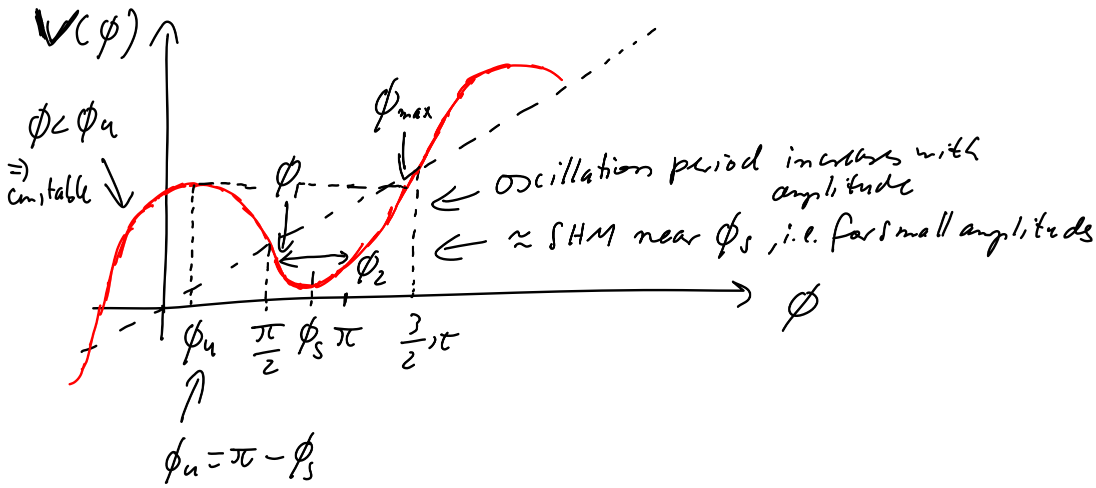
$\Rightarrow$ multiply by $\dot{\phi}$, integrate over time

$$\underbrace{\frac{1}{2} I \dot{\phi}^2}_{E_{kin} = \frac{L^2}{2I}} + \underbrace{mg\ell(\cos\phi + \phi\sin\phi_s)}_{E_{pot} = V(\phi)} = const = H$$

Hamiltonian function

$$E_{pot} = V(\phi) = mgh_1 + mgh_2 = mg\ell\cos\phi + mgr\phi = mg\ell\cos\phi + mg\ell\sin\phi_s\,\phi$$

$\rightsquigarrow$ conservation of energy!



- Trajectories in the $L-\phi$ phase space:

$H = \text{const}$ for trajectory

$$\Rightarrow L = \pm\sqrt{2I\left(H - mg\ell(\cos\phi + \phi\sin\phi_s)\right)}$$

$\Rightarrow$ for small $\Delta\phi = \phi - \phi_s$

$$L = \pm\sqrt{2I\left\{H - mg\ell\cos\phi_s\left(1 - \frac{\Delta\phi^2}{2}\right)\right\}} \Rightarrow \text{ellipse in } L-\phi \text{ space}$$

$\Rightarrow$ at turning points: ϕ_1, ϕ_2 : $L = 0$

$$\cos\phi_{turn} + \phi_{turn}\sin\phi_s = \frac{H}{mg\ell} = \frac{V(\phi_{turn})}{mg\ell}$$

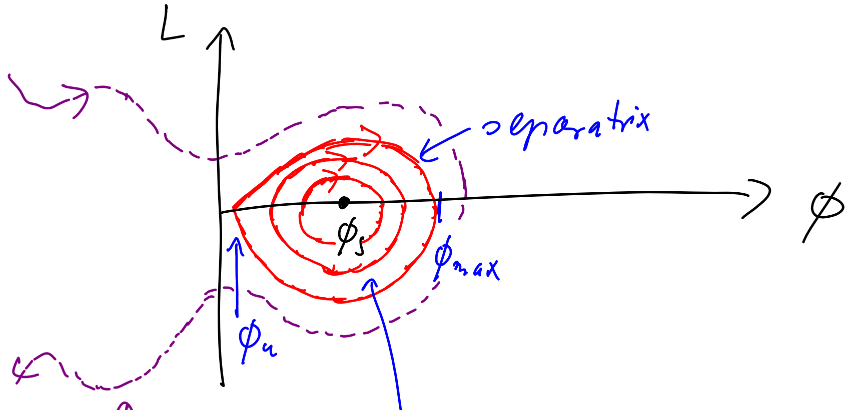
$\Rightarrow$ Separatrix: stable oscillation / trajectory in L-ϕ phase space with largest possible oscillation amplitude:

$$\phi_1 = \phi_u = \pi - \phi_s$$

$$\rightsquigarrow H_{sep} = mgl\{\underbrace{\cos\phi_u}_{-\cos\phi_s} + \underbrace{\phi_u \sin\phi_s}_{(\pi-\phi_s)\sin\phi_s}\}$$

$$\Rightarrow L_{sep} = \pm\sqrt{2Igml\{(\pi-\phi_s-\phi)\sin\phi_s - \cos\phi_s - \cos\phi}$$

Trajectories in the L-ϕ phase space



outside separatrix: unbound motion

region inside separatrix: stable oscillations $\Rightarrow$ bound trajectories inside "bucket"

- back to longitudinal phase space in circ. accelerator:

had: $$\frac{E_0 \beta_0^2}{\omega_0^2 h \eta} \frac{d^2\phi}{dt^2} = \frac{q\hat{V}}{2\pi}(\sin\phi - \sin\phi_0)$$

$\Rightarrow$ multiply by $\dot{\phi}$, integrate over time

$$\frac{1}{2}\frac{E_0\beta_0^2}{\omega_0^2 h\eta}\dot{\phi}^2 + \frac{q\hat{V}}{2\pi}(\cos\phi + \phi\sin\phi_0) = \text{const}$$

$\Rightarrow$ with equ. (2) from lecture 22:

$$\dot{\phi}^2 = \left(h\,\omega_0\,\eta\,\frac{1}{\beta_0^2}\frac{\Delta E}{E_0}\right)^2$$

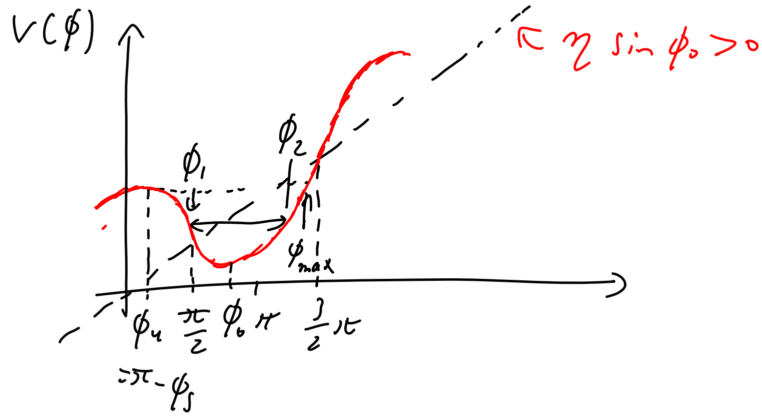
$$\Rightarrow \underset{\text{"kinetic energy"}}{\Delta E^2} + \underbrace{\frac{\beta_0^2 q\hat{V} E_0}{\pi h \eta}(\cos\phi + \phi\sin\phi_0)}_{\text{"potential energy" } V(\phi)} = \text{const} = \underset{\text{hamiltonian function of oscillation}}{H}$$

$\Rightarrow$ local minima/maxima at $\phi = \phi_0$ and $\phi = \pi - \phi_0$

$\Rightarrow$ for small $\Delta\phi = \phi - \phi_0$, particle oscillate in potential well $V(\phi)$ about ϕ_0

if $-\frac{\pi}{2} < \phi_0 < \frac{\pi}{2}$ $(\cos\phi_0 > 0)$ for $\eta < 0$

or if $\frac{\pi}{2} < \phi_0 < \frac{3}{2}\pi$ $(\cos\phi_0 < 0)$ for $\eta > 0$



Potential energy V(Φ)

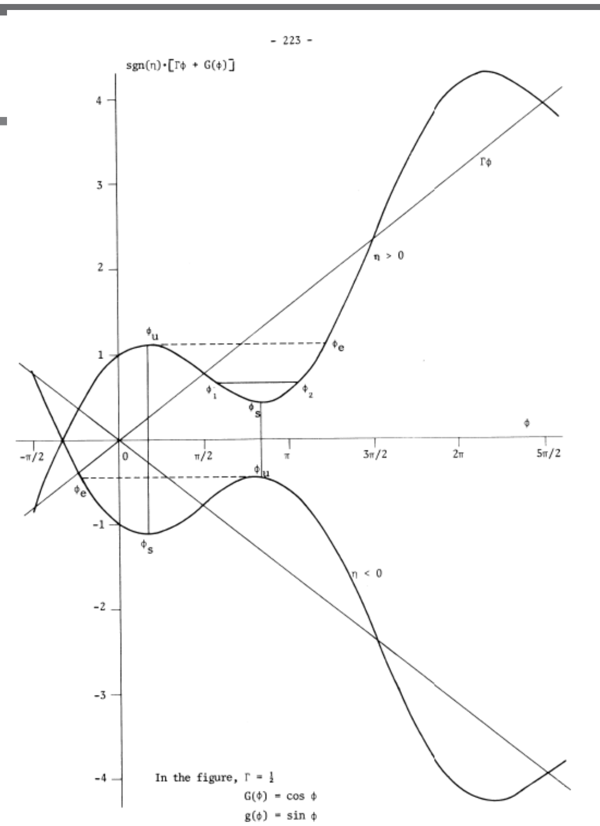


Fig. 3 - Potential energy as a function of φ.

⇒ trajectories in $\Delta E - \phi$ phase space for longitudinal particle motion

$$\Delta E = \pm\sqrt{H - \frac{\beta_0^2 q \hat{V} E_0}{\pi h \eta}\left(\cos\phi + \phi\sin\phi_0\right)}$$

⇒ for small $\Delta\phi = \phi - \phi$ → ellipse

⇒ turning points ϕ_1, ϕ_2 : $\Delta E = 0$

$$\cos\phi_{turn} + \phi_{turn}\sin\phi_0 = \frac{H}{\frac{\beta_0^2 q \hat{V} E_0}{\pi h \eta}}$$

⇒ separatrix = boundary between stable region with synchrotron oscillations and unstable region

for separatrix : $\phi_1 = \phi_u = \pi - \phi_0$

$$\rightsquigarrow H_{sep} = \frac{\beta_0^2 q \hat{V} E_0}{\pi h \eta}\left\{(\pi - \phi_0)\sin\phi_0 - \cos\phi_0\right\}$$

$$\rightsquigarrow \Delta E_{sep} = \pm\sqrt{\frac{\beta_0^2 q \hat{V} E_0}{\pi h \eta}\left\{(\pi - \phi_0 - \phi)\sin\phi_0 - \cos\phi_0 - \cos\phi\right\}}$$

- for $\eta > 0$ case:
  - largest bucket for $\phi_0 = \pi$ $\Rightarrow$ $\phi_u = 0$

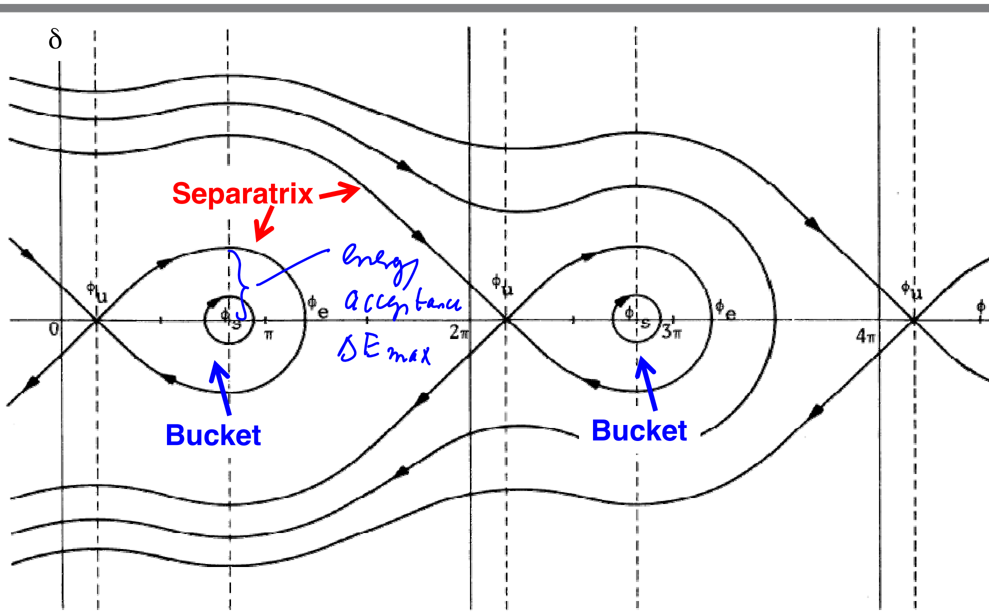
    $\phi_{max} = 2\pi$

    $V = \hat{V} \sin \pi = 0$: operation at constant energy
  - vanishing bucket for $\phi_0 = \frac{\pi}{2}$ $\Rightarrow$ $\phi_u = \pi - \phi_0 = \frac{\pi}{2}$

    $\phi_{max} = \pi/2$

    $V = \hat{V} \cdot 1$ $\Rightarrow$ max energy gain in cavity ("on-crest acceleration")

Longitudinal Phase Space: Trajectories

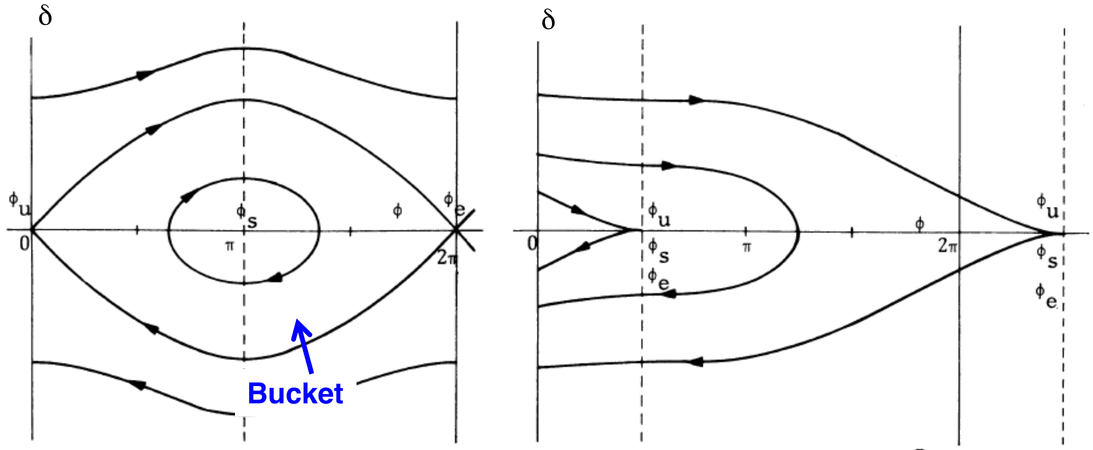


Accelerating bucket $(\frac{\pi}{2} < \phi_s < \pi)$

Trajectories in synchrotron phase space, when $\eta > 0$;
when $\eta < 0$, ϕ_s and ϕ_u are interchanged.

The complete phase space is wrapped around a cylinder $0 \leq \phi \leq 2\pi h$.

Longitudinal Phase Space: Trajectories



Stationary bucket ($\phi_s = \pi$) Vanishing bucket ($\phi_s = \frac{\pi}{2}$)

Trajectories in synchrotron phase space, when $\eta > 0$;
when $\eta < 0$, ϕ_s and ϕ_u are interchanged.
The complete phase space is wrapped around a cylinder $0 \le \phi \le 2\pi h$.

- energy acceptance = $\Delta E_{max} = \Delta E(\phi = \phi_0)$
  (as limited by RF acceleration)

$$\Delta E_{max} = \pm \sqrt{\frac{\beta_0^2 q \hat{V} E_0}{\pi h \eta} \left\{ (\pi - 2\phi_0) \sin\phi_0 - 2\cos\phi_0 \right\}}$$

note: $\Delta E_{max} \propto \sqrt{\hat{V}}$

- full ring has h stable buckets ($h = \omega_{RF}/\omega_0$)

note: dispersion D should be zero at RF cavity!