Lecture 6

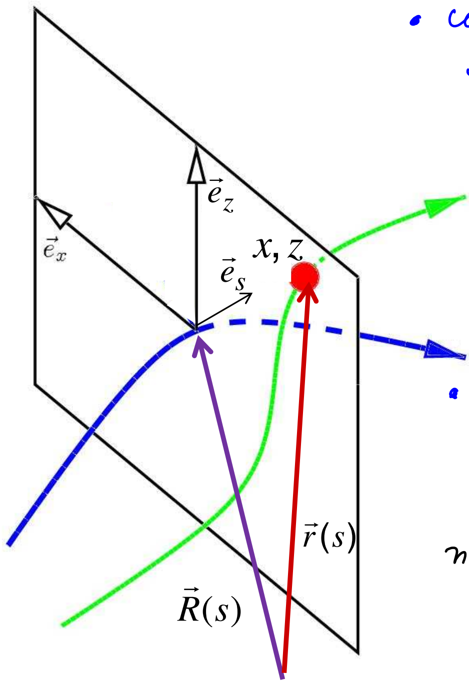
3. Linear transverse beam optics

3.1 Equation of motion

3.2 General solution of the equation of motion

3.1 Equation of motion in linear approximation

A moving (curvi-linear) coordinate system



- use right handed coordinate system (x, z, s) that follows an ideal particle along the ideal path

  ideal path = design orbit

  has straight sections + curved sections with constant curvature

  $$\frac{1}{\rho} = \frac{q}{p} B_{dipole}$$

  straight section: $\rho = \infty \Leftrightarrow 1/\rho = 0$

- particle position vector:

  $$\vec{r} = \vec{R}(s) + x\,\vec{e}_x(s) + z\,\vec{e}_z(s)$$

  arc length along the design orbit

note: $|d\vec{R}| = ds \qquad \vec{e}_s = \frac{d}{ds}\vec{R}(s)$

Equation of Motion (I)

$$\tfrac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a
6 dimensional ODE of 1st order:

$$\left.\begin{aligned} \tfrac{d}{dt}\vec{r} &= \tfrac{1}{m\gamma}\vec{p} = \tfrac{c}{\sqrt{p^2-(mc)^2}}\vec{p} \\ \tfrac{d}{dt}\vec{p} &= \vec{F}(\vec{r}, \vec{p}, t) \end{aligned}\right\} \quad \tfrac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z}, t)\,, \quad \vec{Z} = (\vec{r}, \vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length "s". Using "s" as the independent variable reduces the dimensions to 4.

$$\tfrac{d}{ds}\vec{z} = \vec{f}_z(\vec{z}, s)\,, \quad \vec{z} = (x, y, p_x, p_y)$$

Equation of Motion (II)

Usually one prefers to compute the trajectory as a function of “s” along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy “E” and the time “t” at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\tfrac{d}{ds}\vec{z} = \vec{f}_z(\vec{z}, s)\ , \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

Equation of motion in linear approximation

In the following:

– look at transverse motion only:

$$x,\ z,\quad x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{v_x}{v_s} = \frac{p_x}{p_s} \approx \frac{p_x}{p_0}$$

$$z' = \frac{dz}{ds} = \frac{dz}{dt}\frac{dt}{ds} = \frac{v_z}{v_s} = \frac{p_z}{p_s} \approx \frac{p_z}{p_0}$$

in addition: need $\delta = \frac{E - E_0}{E_0} \approx \frac{p - p_0}{p_0}$ ← design momentum

E_0: design energy; $\approx$: highly relativistic

– assume only linear, constant magnetic fields (transverse), no electric fields

dipole + quad. only!

$$B_x = \pm\frac{p_0}{q} k z \quad \leftarrow \text{quadrupole strength}$$

$$B_z = \frac{p_0}{q}\frac{1}{\rho} \pm \frac{p_0}{q} k x$$

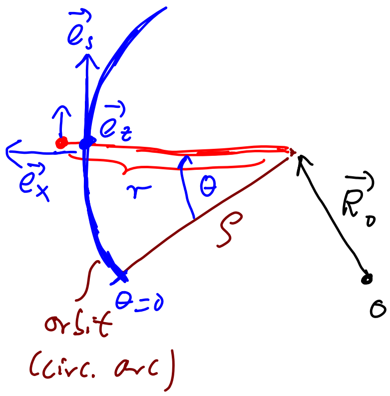
$1/\rho$ = curvature in dipole field

– only consider first order terms in small quantities

⇒ not too small bending radii!

⇒ equ. of motion in linear approximation!

Equation of motion in linear approximation

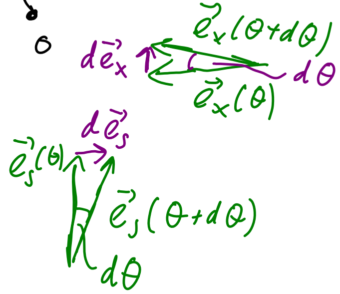


particle position vector:

$$\vec r = \vec R_0 + r\,\vec e_x + z\,\vec e_z$$

$$r = \rho + x$$

for small $d\theta$ (cylindrical coordinates!)



$$d\vec e_x = d\theta\,\vec e_s$$

$$d\vec e_s = -d\theta\,\vec e_x$$

$$d\vec e_z = 0$$

$$\Rightarrow \frac{d\vec v}{dt} = \dot{\vec r} = \dot r\,\vec e_x + r\,\dot{\vec e}_x + \dot z\,\vec e_z = \dot r\,\vec e_x + r\,\dot\theta\,\vec e_s + \dot z\,\vec e_z$$

$\dot{\vec e}_x = d\vec e_x/dt = d\theta/dt\;\vec e_s$

Equation of motion in linear approximation

$\Rightarrow$ for acceleration:

= 0 here (no longitudinal acceleration)

$$\ddot{\vec r} = (\ddot r - r\dot\theta^2)\,\vec e_x + (2\dot r\dot\theta + r\ddot\theta)\,\vec e_s + \ddot z\,\vec e_z$$

- now: $\vec F = \frac{d\vec p}{dt} = \gamma m \ddot{\vec r}$ since γ = const here ($\vec F \perp \vec v$)

$$\Rightarrow m\ddot{\vec r} = q\,\vec v \times \vec B = q\left[(\dot z B_s - r\dot\theta B_z)\,\vec e_x + (\dot r B_z - \dot z B_x)\,\vec e_s + (r\dot\theta B_x - \dot r B_s)\,\vec e_z\right]$$

$\Rightarrow$ assume $B_s = 0$ tangential fields only

$$m(\ddot r - r\dot\theta^2) = -q\,r\,\dot\theta\,B_z(x,z,s)$$

$$m\,\ddot z = q\,r\,\dot\theta\,B_x(x,z,s)$$

$\Rightarrow$ for linear beam optics:

$$B_x = \pm g\,z = \pm\frac{p_0}{q}\,k\,z \qquad B_z = B_0 \pm g\,x = \frac{p_0}{q}\frac{1}{\rho} \pm kx$$

Equation of motion in linear approximation

$\Rightarrow$ with $r = \rho + x$, ρ = const within given dipole magnet

$$m(\ddot{x} - r\dot{\theta}^2) = -r\dot{\theta}\left(p_0 \frac{1}{\rho} \pm p_0 k x\right)$$

$$m\ddot{z} = \pm r\dot{\theta}\, p_0 k z$$

$\Rightarrow$ since $v_s = r\dot{\theta} \gg v_x, v_z$

$$\Rightarrow v_s = r\dot{\theta} \approx |\vec{v}|$$

$\Rightarrow$ also: replace time variable by arc length s along design orbit:

$$s = vt \Rightarrow \quad \ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2 = x'' v^2$$

$$\ddot{z} = z'' v^2$$

$$\Rightarrow \quad x'' \pm \frac{p_0}{p} k x = \frac{1}{r} - \frac{p_0}{p}\frac{1}{\rho} \qquad z'' \mp \frac{p_0}{p} k z = 0$$

Equation of motion in linear approximation

particle momentum: $p = p_0(1+\delta)$ $\qquad \delta = \frac{\Delta p}{p_0} \leftarrow$ momentum error of particle

$\Rightarrow$ to first order approximation:

$$\text{for } \delta \ll 1: \quad \frac{1}{p} = \frac{1}{p_0(1+\delta)} \approx \frac{1}{p_0}(1-\delta)$$

$$\text{for } \rho \gg x: \quad \frac{1}{r} = \frac{1}{\rho(1+x/\rho)} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

$\Rightarrow$ to first order in x, z, δ:

$$\boxed{\begin{aligned} & x'' + \left(\pm k + \frac{1}{\rho^2}\right)x = \frac{\delta}{\rho} \quad (+\, O(2) + O(3) + \dots) \\ & z'' \mp k z = 0 \quad (+\, O(2) + O(3) + \dots) \end{aligned}}$$

equ. of transverse motion in linear beam optics

Equation of motion in linear approximation

- $1/\rho^2$ term: "weak focusing" in dipole magnet
  - in dipole: $K = 0$
  - in quadrupole: $1/\rho = 0$
  - in drift: $K = 0$, $1/\rho = 0$

Note: all terms on the right-hand side can be treated as small perturbations!

3.2 General solution of the equation of motion

linear, unperturbed equation of motion

have homogeneous differential equation:

(eq 1) $$u'' + \underbrace{\mathcal{K}(s)}_{\text{"}\omega^2\text{"}}\, u = 0$$ } like harmonic oscillator with time-dep./s-dep. frequency

where u stands for x or z

and $\mathcal{K} = \pm k + \frac{1}{\rho^2}$ or $\mathcal{K} = \mp k$ for given magnet

⇒ principal solutions (there should be two linearly indep. solutions!) for $\mathcal{K} = \text{const}$:

for $\mathcal{K} > 0$ $\quad C(s) = \cos(\sqrt{\mathcal{K}}\, s)$ and $S(s) = \frac{1}{\sqrt{\mathcal{K}}} \sin(\sqrt{\mathcal{K}}\, s)$

for $\mathcal{K} < 0$ $\quad C(s) = \cosh(\sqrt{|\mathcal{K}|}\, s)$ and $S(s) = \frac{1}{\sqrt{|\mathcal{K}|}} \sinh(\sqrt{|\mathcal{K}|}\, s)$

with initial conditions:

$$C(0) = 1 \qquad C'(0) = \frac{dC}{ds}\Big|_0 = 0$$

$$S(0) = 0 \qquad S'(0) = \frac{dS}{ds}\Big|_0 = 1 \qquad \text{(equ. 2)}$$

⇒ any arbitrary solution $u(s)$ can be expressed as a linear combination of these two principal solutions:

$$\left.\begin{aligned} u(s) &= C(s)\, u_0 + S(s)\, u_0' \\ u'(s) &= C'(s)\, u_0 + S'(s)\, u_0' \end{aligned}\right\} \text{(equ. 3)}$$

where u_0 and u_0' are the initial values of $u(s)$ and $u'(s)$ at $s = 0$

- general beam line with several magnets:

$$\mathcal{K} = \mathcal{K}(s)$$

$\Rightarrow$ principle solutions (so-called sine like and cosine like solutions) can be found with initial conditions as above for $\mathcal{K} = \text{const}$ case!

[$u'' + \mathcal{K}(s)\, u = 0$ insert ansatz (equ 3)

$\Rightarrow$ $[S''(s) + \mathcal{K}(s)\, S(s)]\, u_0 + [C''(s) + \mathcal{K}(s)\, C(s)]\, u_0' = 0$

for any initial condition (u_0, u_0')

$$\Rightarrow \quad S''(s) + \mathcal{K}(s)\, S(s) = 0$$

$$C''(s) + \mathcal{K}(s)\, C(s) = 0$$

i.e. general solution can be written as a sum of two linearly independent solutions $C(s)$ and $S(s)$]

Matrix formulation:

now: solution (eq. 3) of equ. of motion (1) in matrix formulation

$$\underbrace{\begin{bmatrix} u(s) \\ u'(s) \end{bmatrix}}_{\substack{\text{at exit}\\ \text{of beamline/}\\ \text{magnet}}} = \underbrace{\begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix}}_{\substack{\text{transformation}\\ \text{matrix of given}\\ \text{beamline/ magnet}}} \underbrace{\begin{bmatrix} u_0 \\ u_0' \end{bmatrix}}_{\substack{\text{initial}\\ \text{condition}}}$$

$\Rightarrow$ from $\mathcal{K} = \text{const}$ solutions: can obtain transformation matrix for each individual beam line element (magnet)

$\Rightarrow$ by repeated matrix multiplication, can follow particle trajectory along a complicated beam line!