Lecture 7

3. Linear transverse beam optics

3.2 General solution of the equation of motion

3.3 Building blocks for beam transport lines

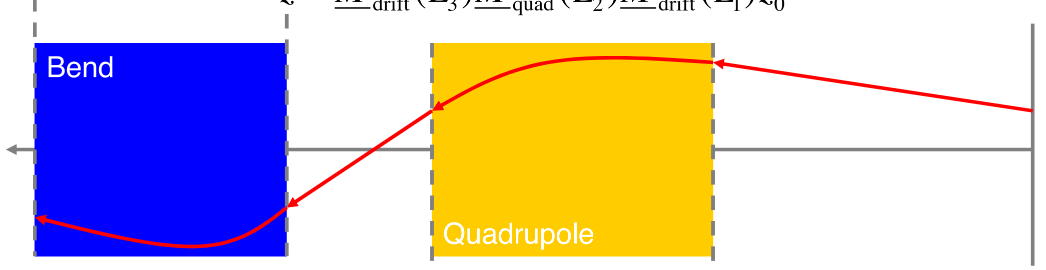
Matrix Formulation

$$\vec{z}_s = \underline{M}(s)\vec{z}_0 \qquad \begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} \begin{bmatrix} u_0 \\ u'_0 \end{bmatrix}$$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$

$$\vec{z} = \underline{M}_{\text{bend}}(L_4)\underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$



$$\vec{z} = \underline{M}_{\text{drift}}(L_1)\vec{z}_0 \qquad \vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

Properties of the transformation matrix (I)

Consider a liner homogeneous differential equation of second order: $u'' + v(s)\, u' + w(s) u = 0$

$\Rightarrow$ rules describing the propertis of the solution

(a) there is only one solution which meets the initial conditions $u(s_0) = u_0$, $u'(s_0) = u_0'$ at $s = s_0$

(b) $c \cdot u(s)$ is also a solution if $u(s)$ is a solution and $c = $ cont

(c) if $u_1(s)$ and $u_2(s)$ are two linearly independent solutions, any linear combination of these two is also a solution.

Properties of the transformation matrix (II)

(d) for two linearly independent solutions $u_1(s)$ and $u_2(s)$

Wronskian determinant:

$$W = \begin{vmatrix} u_1(s) & u_2(s) \\ u_1'(s) & u_2'(s) \end{vmatrix} = u_1 u_2' - u_2 u_1' \neq 0$$

with
$$u_1'' + v(s) u_1' + w(s) u_1 = 0 \quad | -u_2$$
$$u_2'' + v(s) u_2' + w(s) u_2 = 0 \quad | \; u_1$$

Combine: $(u_1 u_2'' - u_2 u_1'') + v(s)(u_1 u_2' - u_2' u_1) = 0$

$$\Rightarrow \frac{dW}{ds} = u_1 u_2'' - u_2 u_1'' \overset{\checkmark}{=} -v(s)\, W$$

integrate: $W(s) = W_0 \, e^{-\int_{s_0}^{s} v(s)\, ds}$

Properties of the transformation matrix (III)

Conclusion: have $\mathcal{V}(s) = 0$ (as long as we don't include dissipating forces like acceleration or synchrotron radiation)

$$\Rightarrow \boxed{W(s) = W_0 = \text{const}} \qquad \left(\frac{dW}{ds} = 0 \right)$$

from initial conditions:

$$C(0) = 1 \qquad C'(0) = 0 \qquad S(0) = 0 \qquad S'(0) = 1$$

$$\Rightarrow \underline{\underline{W}} = \begin{vmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{vmatrix} = W_0 = C_0 S_0' - C_0' S_0 = \underline{\underline{1}}$$

$$(\text{if } \mathcal{V}(s) = 0)$$

Solution of the inhomogeneous equation of motion (I)

equation of motion: $u'' + K(s)\, u = p(s)$

general solution:

$$u(s) = \underbrace{C(s)\, a + S(s)\, b}_{\text{general solution of the homogeneous equation with } a, b \text{ determined by initial parameters}} + \underbrace{P(s)}_{\text{a particular solution of the inhomogeneous equation}}$$

– particular solution $P(s)$

can be found from $\underline{P}(s) = \int_0^s p(\tilde{s})\, \underbrace{G(s, \tilde{s})}_{\text{suitable Green's function}}\, d\tilde{s}$

suitable Green's function; can be constructed from the principle solutions of the homog. eqn

Solution of the inhomogeneous equation of motion (II)

$$G(s,\tilde{s}) = S_1(s)\,C_1(\tilde{s}) - C_1(s)\,S_1(\tilde{s})$$

$\Rightarrow$ for the particular solution:

$$P(s) = S_1(s)\int_0^s p(\tilde{s})\,C_1(\tilde{s})\,d\tilde{s} - C_1(s)\int_0^s p(\tilde{s})\,S(\tilde{s})\,d\tilde{s}$$

Proof: need $P''(s)$

$$P'(s) = S_1'(s)\int_0^s p(\tilde{s})\,C_1(\tilde{s})\,d\tilde{s} + S_1(s)\,C_1(s)\,p(s) - C_1'(s)\int_0^s p(\tilde{s})\,S_1(\tilde{s})\,d\tilde{s} - C_1(s)\,S_1(s)\,p(s),$$

$$\Rightarrow P''(s) = S_1''(s)\int_0^s p(\tilde{s})\,C_1(\tilde{s})\,d\tilde{s} + S_1'(s)\,C_1(s)\,p(s) - C_1''(s)\int_0^s p(\tilde{s})\,S(\tilde{s})\,d\tilde{s} - C_1'(s)\,S(s)\,p(s) = p(s) + S''(s)\int_0^s p(\tilde{s})\,C_1(\tilde{s})\,d\tilde{s}, - C''(s)\int_0^s p(\tilde{s})\,S_1(\tilde{s})\,d\tilde{s}$$

$$|w| = CS' - SC' = 1$$

Solution of the inhomogeneous equation of motion (III)

with $\quad S'' + \mathcal{K}S = 0 \qquad C'' + \mathcal{K}C = 0$

$$\Rightarrow P''(s) + \mathcal{K}(s)\,P(s) = p(s)$$

$\Rightarrow$ $P(s)$ is a particular solution of the inhomogeneous equation q.e.d.

Dispersion Function (I)

most important example: perturbation term from momentum error: $\delta = 1 - P/P_0 << 1$

(chromatic error)

$$u'' + \mathcal{K}(s)\, u = \frac{1}{\rho(s)}\, \delta$$

$\Rightarrow$ deviation in path from path of particle with nominal energy

$$u(s) = C(s)\, a + S(s)\, b + P(s)$$

now, since $P(s) \propto \delta \Rightarrow P(s) \propto \delta$

$\Rightarrow$ define normalized dispersion function:

$$D(s) = \frac{P(s)}{\delta}$$

Dispersion Function (II)

which is a particular solution of the inhomogen. equation: $D''(s) + \mathcal{K}(s)\, D(s) = \frac{1}{\rho(s)}$

from above:

$$\underline{\underline{D(s)}} = S(s) \int_0^s \frac{1}{\rho(\tilde{s})}\, C(\tilde{s})\, d\tilde{s} - C(s) \int_0^s \frac{1}{\rho(\tilde{s})}\, S(\tilde{s})\, d\tilde{s}$$

General solution with chromatic correction

$$u(s) = C(s)u(s_0) + S(s)u'(s_0) + \delta D(s)$$

$$u'(s) = C'(s)u(s_0) + S'(s)u'(s_0) + \delta D'(s)$$

with initial conditions at $s = s_0$

$C(s_0) = 1 \quad C'(s_0) = 0 \quad S(s_0) = 0 \quad S'(s_0) = 1$

$D(s_0) = 0 \quad D'(s_0) = 0$

$$\begin{bmatrix} u(s) \\ u'(s) \\ \delta \end{bmatrix} = \begin{bmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u(s_0) \\ u'(s_0) \\ \delta \end{bmatrix}$$

3.3 Building blocks for beam transport lines

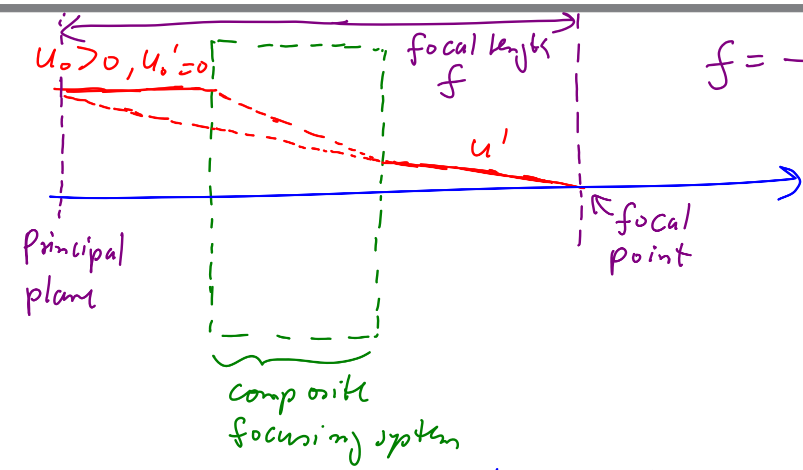
general focusing

zero dispersion

first order achromat

first order isochronous

(a) General focusing (δ=0)



$$f = -\frac{u_0}{u'(s)}$$

⇒ for $u_0' = 0$ $\quad u'(s) = C'(s)\,u_0\,,\quad u(s) = C(s)\,u_0$

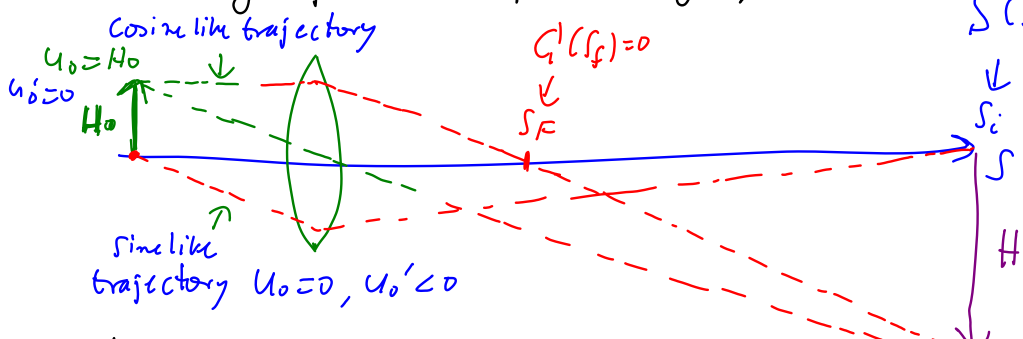
⇒ at focal point: $u(s_f) = 0 \Rightarrow C'(s_f) = 0$

⇒ focal length of focusing system: $f^{-1} = C'(s)$

(large C' → strong focusing / defoc.)

More on general focusing (δ=0)

⇒ Magnification of focusing system:



⇒ beam size H_i at image point: $H_i = |C(s_i)| \cdot H_0$

$u(s_i) = C(s_i)\,H_0$

⇒ Magnification: $M = |C(s_i)|$

(large $C(s_i)$ ⇒ large magnification)

(b) Zero Dispersion point: D(s)=0 at s=s$_d$

had: $D(s) = S(s)\int_0^s \frac{1}{\rho(\tilde{s})} C'(\tilde{s})\,d\tilde{s} - C'(s)\int_0^s \frac{1}{\rho(\tilde{s})} S'(\tilde{s})\,d\tilde{s}$

$\Rightarrow$ only dipole fields can change dispersion
($1/\rho = 0$ otherwise...) in linear approximation

define: $I_c(s) = \int_0^s \frac{1}{\rho(\tilde{s})} C'(\tilde{s})\,d\tilde{s}$

$I_s(s) = \int_0^s \frac{1}{\rho(\tilde{s})} S'(\tilde{s})\,d\tilde{s}$

$\Rightarrow$ for $D(s) = 0$ at $s = s_d$, need:

$$\left.\frac{S'(s_d)}{C'(s_d)} = \frac{I_s(s_d)}{I_c(s_d)}\right\}$$ adjust focusing structure accordingly

(c) First order achromatic lattice

require: $D(s_d) = 0$ and $D'(s_d) = 0$

$\Rightarrow$ $D(s > s_d) = 0$ downstream of s_d up to next dipole magnet

$\Rightarrow$ position and slope of particle independent of energy at end of achromat!

$$D(s_d) = S'(s_d)\,I_c(s_d) - C'(s_d)\,I_s(s_d) = 0$$

$$D'(s_d) = S''(s_d)\,I_c(s_d) - C''(s_d)\,I_s(s_d) = 0$$

$\Rightarrow$ solve for I_c and I_s:

$$[C'(s_d)\,S''(s_d) - C''(s_d)\,S'(s_d)]\,I_c(s_d) = 0$$

$$[\underbrace{C'(s_d)\,S''(s_d) - C''(s_d)\,S'(s_d)}_{=1 \text{ since } |W|=1}]\,I_s(s_d) = 0$$

First order achromat

$\Rightarrow$ for $D(s_d) = 0$ and $D'(s_d) = 0$

conditions:

$$I_c(s_d) = \int_0^{s_d} \frac{1}{\rho(\tilde{s})} C'(\tilde{s})\, d\tilde{s} = 0$$

$$I_s(s_d) = \int_0^{s_d} \frac{1}{\rho(\tilde{s})} S'(\tilde{s})\, d\tilde{s} = 0$$

for first order achromat