# Lecture 8

## 3. Linear transverse beam optics

### 3.3 Building blocks for beam transport lines

Isochronous systems

### 3.4 Transformation matrices of accelerator magnets



# (d) Isochronous systems

require: time of flight through beam line same for all particles, even if $\delta \neq 0$

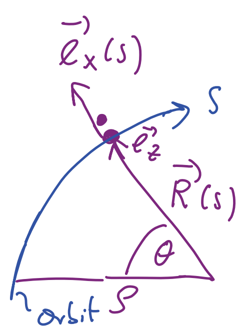
– highly relativistic particles: $\gamma \gg 1$

same time of flight $\Leftrightarrow$ same path length

– path length $L$:

for arbitrary particle: $L = \int_0^{s=L_0} dr = \int_0^{s=L_0} \frac{dr}{ds}\, ds$

$dr$: path length element of particle trajectory




particle position:

$\vec{r} = \vec{R}(s) + x\,\vec{e}_x + z\,\vec{e}_z$

assume $\vec{e}_z = \text{const}$ (only horizontal deflection of the orbit)

$$\Rightarrow \quad \frac{d\vec{r}}{ds} = 1\cdot \vec{e}_s + x'\vec{e}_x + x\frac{\vec{e}_s}{\rho} + z'\vec{e}_z$$

$\uparrow |d\vec{R}| = ds$ $\qquad\qquad$ $\uparrow \frac{d\vec{e}_x}{ds} = \frac{d\vec{e}_x}{d\theta}\frac{d\theta}{ds} = \frac{\vec{e}_s}{\rho}$

$$\Rightarrow \quad \left|\frac{d\vec{r}}{ds}\right| = \frac{dr}{ds} = \sqrt{x'^2 + z'^2 + (1+x/\rho)^2}$$

$$\Rightarrow \quad L = \int_0^{L_0} \sqrt{x'^2 + y'^2 + \left(1+\frac{x}{\rho}\right)^2}\, ds \approx \int_0^{L_0} \left[1 + \frac{x(s)}{\rho(s)}\right] ds + O(2) + O(3)$$

to first order in $x' \ll 1$, $z' \ll 1$, $\frac{x}{\rho} \ll 1$

now: $x(s) = C(s)\,x_0 + S(s)\,x_0' + D(s)\,\delta$

$$\Rightarrow (L - L_0) \approx x_0 \int_0^{L_0} \frac{1}{\rho(s)} C(s)\,ds + x_0' \int_0^{L_0} \frac{1}{\rho(s)} S(s)\,ds + \delta \int_0^{L_0} \frac{1}{\rho(s)} D(s)\,ds$$



$\Rightarrow$ for first order isochronous beam line ($\gamma \gg 1$)
need $(L - L_0) = 0$ for any initial condition $x_0, x_0', \delta$

$\Rightarrow$ beam line needs to be an first order achromat ($I_c = 0$, $I_s = 0$) with the additional condition:

$$I_d(L_0) = \int_0^{L_0} \frac{1}{\rho(s)} D(s)\,ds = 0$$

# 3.4 Transformation matrices of accelerator magnets

Drift
Dipole
Edge focusing
Quadrupole
Combined function magnet
Thin lens approximation



→ Matrix formulation in Linear beam optics

$$\begin{bmatrix} u(s) \\ u'(s) \\ \delta \end{bmatrix} = \begin{bmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u(s_0) \\ u'(s_0) \\ \delta \end{bmatrix}$$

for $\mathcal{K} = k + \frac{1}{\rho^2} = \text{const}$

if $\mathcal{K} > 0$ $\quad C = \cos(\sqrt{\mathcal{K}}\, s) \quad S(s) = \frac{1}{\sqrt{\mathcal{K}}} \sin(\sqrt{\mathcal{K}}\, s)$

if $\mathcal{K} < 0$ $\quad C = \cosh(\sqrt{\mathcal{K}}\, s) \quad S(s) = \frac{1}{\sqrt{\mathcal{K}}} \sinh(\sqrt{\mathcal{K}}\, s)$

$$D(s) = S(s) \int_0 \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C(s) \int_0 \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

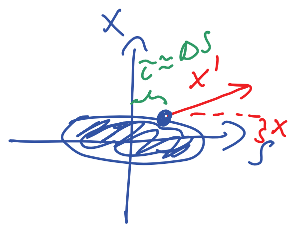
$$= S(s)\, I_c(s) - C(s)\, I_s(s)$$

# 6D Phase space

$\rightarrow$ path length: $(L - L_0) = x_0 I_c + x_0' I_s + \delta I_d$

Now: combine x, z, longitudinal motion

$\Rightarrow$ 6D Phase space vector of particle:




$$\vec{X} = \begin{bmatrix} x \\ x' \\ z \\ z' \\ \tau \\ \delta \end{bmatrix}$$

$$x' = \frac{dx}{ds} \approx \frac{p_x}{p_0}$$

$$\delta = \frac{\Delta E}{E_0} \approx \frac{\Delta p}{p_0}$$

$$\tau = (t_0 - t)\frac{c^2}{v_0}$$

$t_0$: reference time at which bunch center is at "s"



# 6D matrix formulation

$\Rightarrow$ 6 D Matrix formulation:

$$\vec{X}_s = \underline{M}_6 \vec{X}_0 = \begin{bmatrix} \underline{M}_{2,x} & \underline{O} & \begin{matrix} 0 & D \\ 0 & D' \end{matrix} \\ \underline{O} & \underline{M}_{2,z} & \underline{O} \\ \begin{matrix} \vec{T}^T \\ 0 \quad 0 \end{matrix} & \underline{O} & \begin{matrix} 1 & M_{56} \\ 0 & 1 \end{matrix} \end{bmatrix} \vec{X}_0$$

assume $\delta = \text{const}$ here

y-motion does not change path length to first order (assume $1/\rho_y = 0$)

$$\underline{M}_2 = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\tau = (t_0 - t)\frac{c^2}{v_0} = \left(\frac{L_0}{v_0} - \frac{L}{v}\right)\frac{c^2}{v_0} \approx \left[-x_0 I_c - x_0' I_s - \delta I_d\right]$$

$\Rightarrow \vec{T}^T = [-I_c \quad -I_s]$ $\quad M_{56} = -I_d$

## (a) Drift: length $\ell$

$$\left.\begin{aligned} K_x &= k + \frac{1}{\rho^2} = 0 \\ K_z &= -k = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} C_{x,z} &= 1 & S_{x,z} &= 1\cdot\ell \\ C'_{x,z} &= 0 & S'_{x,z} &= 1 \end{aligned}$$

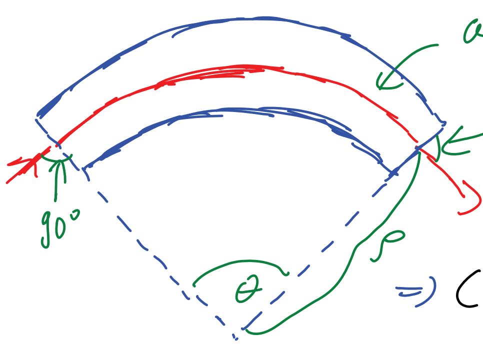
$$\frac{1}{\rho} = 0 \Rightarrow I_c = I_s = I_d = 0 \quad \Rightarrow D = 0,\ D' = 0,\ M_{56} = 0$$

$$\vec{T}^T = [0, 0]$$

$$\Rightarrow \underline{M}_{6,\text{drift}} = \begin{bmatrix} 1 & \ell & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ell & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



## Dipole sector Magnet: horizontal with arc length $\ell$




arc length $\ell$

$90° \Rightarrow K$ does not depend on transverse offset $x$!

$$K_x = \frac{1}{\rho^2} > 0 \qquad K_z = 0$$

$$\Rightarrow C_x = \cos(\ell/\rho) \qquad S_x = \rho \sin(\ell/\rho)$$

$$\left.\begin{aligned} C_z &= 1 \\ S_z &= 1\cdot\ell \end{aligned}\right\} \text{vertical drift}$$

$$D_x = S_x \int_0^\ell \frac{C_x}{\rho}\, d\tilde{s} - C_x \int_0^\ell \frac{S_x}{\rho}\, d\tilde{s}$$

$$= \rho \sin(\ell/\rho) \sin(\ell/\rho) + \cos\left(\frac{\ell}{\rho}\right) \rho \left(\cos\left(\frac{\ell}{\rho}\right) - 1\right)$$

$$= \rho\left(1 - \cos(\ell/\rho)\right)$$

$$D_z = 0 \quad (1/\rho_z = 0)$$

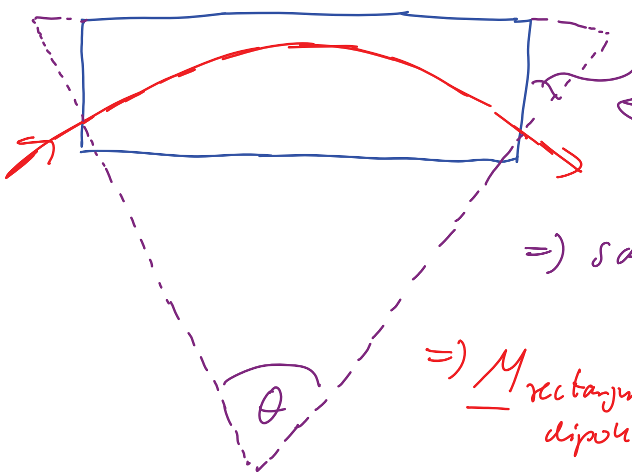
$$M_{56} = -I_d = -\int_0^{\ell} \frac{1}{\rho} \underbrace{\rho\left(1-\cos\left(\frac{\ell}{\rho}\right)\right)}_{D} ds = \rho \sin\left(\frac{\ell}{\rho}\right) - \ell$$

$$\vec{T}^T = \left[-I_c \quad -I_s\right] = \left[-\sin\left(\frac{\ell}{\rho}\right) \quad -\rho\left(1-\cos\left(\frac{\ell}{\rho}\right)\right)\right]$$

$$\Rightarrow M_{6,\,\text{sector dipole}} = \begin{bmatrix} \cos(\ell/\rho) & \rho\sin(\ell/\rho) & 0 & 0 & 0 & \rho(1-\cos(\ell/\rho)) \\ -\frac{1}{\rho}\sin(\ell/\rho) & \cos(\ell/\rho) & 0 & 0 & 0 & \sin(\ell/\rho) \\ 0 & 0 & 1 & \ell & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin(\ell/\rho) & \rho(\cos(\frac{\ell}{\rho})-1) & 0 & 0 & 1 & \rho\sin(\frac{\ell}{\rho})-\ell \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



## (c) Rectangular dipole magnet / edge focusing:



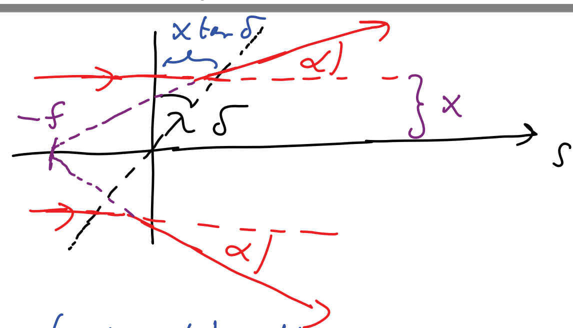

$\delta = \theta/2$

← end plates are not ⊥ to the trajectory!

⇒ same as edge + sector dipole + edge

$$\Rightarrow \underline{M}_{\substack{\text{rectangular}\\ \text{dipole}}} = \underline{M}_{\substack{\text{edge}\\ \delta=\theta/2}} \, \underline{M}_{\substack{\text{sector}\\ \text{dipole}}} \, \underline{M}_{\substack{\text{edge}\\ \delta=\theta/2}}$$

# Edge, horizontally:



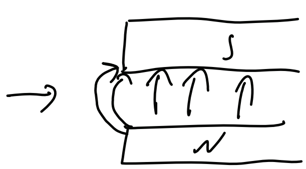

horizontal deflection angle: $\alpha = \dfrac{x \tan\delta}{\rho}$

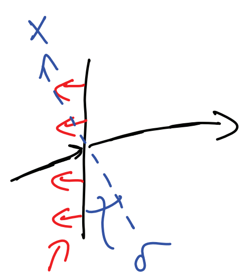
$\Rightarrow$ horizontal defocussing: (for $\delta > 0$)

with focal length: $\left|\dfrac{1}{f}\right| = \dfrac{\alpha}{x} = \dfrac{\tan\delta}{\rho}$



# Edge, vertically:




$B_{horiz} \Rightarrow B_x \neq 0$ if $\delta \neq 0$ and $z \neq 0$

$\Rightarrow$ fringe fields have have horizontal field components

$\Rightarrow$ vertically focusing with same $\dfrac{1}{f} = \dfrac{\tan\delta}{\rho}$ as horizontally!

$\Rightarrow$ edge acts like a quadrupole with: $\dfrac{1}{f} = kl = \dfrac{\tan\delta}{\rho}$

# Edge Focusing

$$\Rightarrow \underline{M}_{edge} = \begin{bmatrix} \begin{matrix} 1 & 0 \\ \frac{\tan\delta}{\rho} & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & 0 \\ -\frac{\tan\delta}{\rho} & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix}$$