Lecture 9

3. Linear transverse beam optics

3.4 Transformation matrices of accelerator magnets

Quadrupole
Combined function magnet
Thin lens approximation

3.5 Momentum compaction Factor

3.6 Twiss parameters (α, β, γ, ψ)

d) Quadrupole: of length ℓ

d₁) horizontal defocusing, vertical focusing

$1/\rho = 0 \;,\; \mathcal{K}_x = -k < 0 \qquad \mathcal{K}_z = k > 0$

$\Rightarrow C_x = \cosh(\sqrt{k}\,\ell) \qquad S_x = \frac{1}{\sqrt{k}} \sinh(\sqrt{k}\,\ell)$

$C_z = \cos(\sqrt{k}\,\ell) \qquad S_z = \frac{1}{\sqrt{k}} \sin(\sqrt{k}\,\ell)$

$D_x = 0 \;,\; D_z = 0 \quad M_{56} = 0 \quad \vec{T} = \vec{0}$

$$\Rightarrow M_{\text{quad vert. foc.}} = \begin{bmatrix} \cosh(\sqrt{k}\,\ell) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,\ell) & \underline{0} & \underline{0} \\ \sqrt{k}\sinh(\sqrt{k}\,\ell) & \cosh(\sqrt{k}\,\ell) & & \\ \underline{0} & & \cos(\sqrt{k}\,\ell) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,\ell) & \underline{0} \\ & & -\sqrt{k}\sin(\sqrt{k}\,\ell) & \cos(\sqrt{k}\,\ell) & \\ \underline{0} & & \underline{0} & & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \end{bmatrix}$$

d2) horizontal focusing, vertical defocusing:

$1/\rho = 0 \qquad \mathcal{K}_x = k > 0 \qquad \mathcal{K}_z = -k < 0$

replace $\quad C_x \leftrightarrow C_z$

$\qquad\qquad S_x \leftrightarrow S_z$

$$M_{quad} = \begin{bmatrix} \cos(\sqrt{k}\,\ell) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,\ell) & \underline{0} & \underline{0} \\ -\sqrt{k}\sin\sqrt{k}\,\ell & \cos\sqrt{k}\,\ell & & \\ \underline{0} & & \cosh\sqrt{k}\,\ell & \frac{1}{\sqrt{k}}\sinh\sqrt{k}\,\ell & \underline{0} \\ & & \sqrt{k}\sinh\sqrt{k}\,\ell & \cosh\sqrt{k}\,\ell & \\ \underline{0} & & \underline{0} & & \underline{1} \end{bmatrix}$$

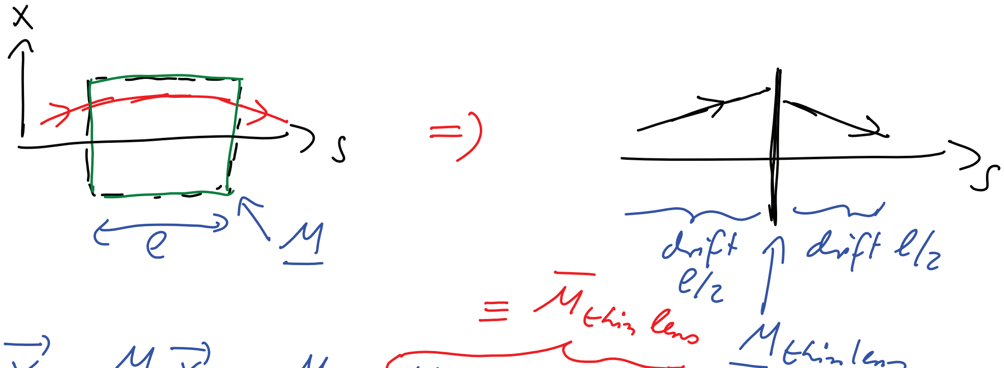
horiz. foc.

e) combined function bend:

$\mathcal{K}_x = \pm k + \frac{1}{\rho^2} \qquad \mathcal{K}_z = \mp k$

$\Rightarrow$ see homework

f) thin lens approximation



$$\vec{X} = \underline{M}\,\vec{X}_0 = \underline{M}_{\substack{\text{drift}\\ l/2}} \overbrace{\underline{M}_{\substack{\text{drift}\\ -l/2}}\,\underline{M}\,\underline{M}_{\substack{\text{drift}\\ -l/2}}}^{\equiv\, \underline{M}_{\text{thin lens}}} \underline{M}_{\substack{\text{drift}\\ l/2}}\,\vec{X}_0$$

for thin drift:

$$\underline{M}_{\text{thin drift}} = \underline{M}_{\substack{\text{drift}\\ -l/2}}\,\underline{M}_{\substack{\text{drift}\\ l}}\,\underline{M}_{\substack{\text{drift}\\ -l/2}} = 1$$

f$_1$) thin, weak lens dipole: for $l/\rho \ll 1$

$$\underline{M}_{\substack{\text{thin}\\ \text{bent}}} = \underline{M}_{\substack{\text{drift}\\ -l/2}}\,\underline{M}_{\text{bend}}\,\underline{M}_{\substack{\text{drift}\\ -l/2}} \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -l/\rho^2 & 1 & 0 & 0 & 0 & l/\rho \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -l/\rho & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

f$_2$) thin, weak lens quadrupole: $\sqrt{k}\,l \ll 1$, $kl = 1/f$

$$\underline{M}_{\substack{\text{thin}\\ \text{quad}\\ \text{vert}\\ \text{foc.}}} = \underline{M}_{\substack{\text{drift}\\ -l/2}}\,\underline{M}_{\substack{\text{quad}\\ \text{vert}\\ \text{foc.}}}\,\underline{M}_{\substack{\text{drift}\\ -l/2}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ kl & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -kl & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

f3) thin, weak combined function bend → see HW

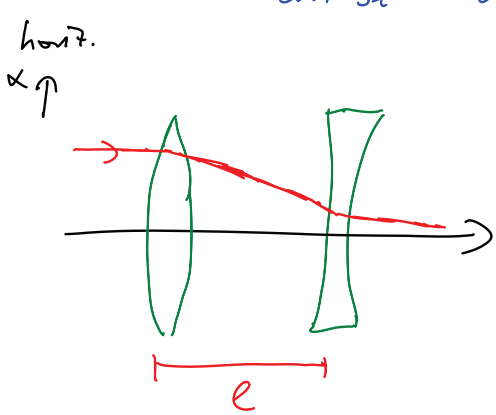
<u>Note:</u> in all cases (even for thin lens approximation)

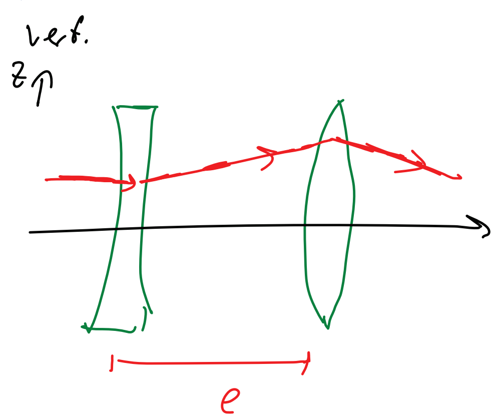
$$\det \underline{M} = 1 \iff \text{Liouville's Theorem of phase space conservation}$$

- simple Taylor expansion for small l/ρ (dipole) or small $\sqrt{k}\, l$ (quadrupole) would <u>not</u> fulfill $\det \underline{M} = 1$!

f4) <u>Quadrupole doublet:</u>

focusing quadrupole + drift + defocusing quadrupole
⇒ both horizontal and vertical focusing can be achieved!





$\Rightarrow$ trajectories entering parallel to axis have larger amplitude in the focusing than in the defocusing lens $\Rightarrow$ overall focusing

$\Rightarrow$ in thin lens approximation and $|f_1| = |f_2|$

$$\underline{M}_{2,x,\,\text{doublet}} = \underline{M}_{2,x,\,\text{def}}\; M_{2,x,\,\text{drift } \ell}\; M_{2,x,\,\text{foc}}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1-\ell/f & \ell \\ -\ell/f^2 & 1+\frac{\ell}{f} \end{pmatrix}$$

Similar:

$$\underline{M}_{2,z,\,\text{doublet}} = \begin{pmatrix} 1+\ell/f & \ell \\ -\ell/f^2 & 1-\ell/f \end{pmatrix}$$

$\Rightarrow$ focusing in both planes with overall refractive power : $M_{21} = C' = -\ell/f^2$

$\Rightarrow$ effective focal length:

$$f_{\text{doublet}} = f^2/\ell$$

g) Accelerating Section

→ consider section of length ℓ with $E_s = \text{const}$

⇒ for $\gamma \gg 1$

$$\frac{dp}{ds} = \frac{q}{c} E_s = \text{const}$$

⇒ momentum of particle

$$p(s) = p_0 + \frac{q}{c} E_s \cdot s$$

⇒ for transverse motion: $(u = x \text{ or } z)$

$$\frac{d}{ds} p_u = \frac{d}{ds}\Big(\underbrace{\frac{p(s)}{c}}_{\simeq \gamma m} v_u \Big) = \frac{d}{ds}\Big(p(s) \frac{du}{ds} \Big) = 0$$

not constant here!

⇒ integrate: $p(s) \frac{du}{ds} = \text{const} = p_0 u_0'$

⇒ $$\frac{du}{ds} = u'(s) = \frac{p_0 u_0'}{p_0 + \frac{q}{c} E_s \cdot s}$$

⇒ integrate:

$$u(s) = u_0 + u_0' \frac{c p_0}{q E_s} \ln\left(1 + \frac{q E_s}{c p_0} s\right)$$

$\Rightarrow$ for $s = \ell$ = length of acc. section

with Δp = momentum gain $= \frac{q}{c} E_s \ell$

$$\underline{M}_{\substack{acc \\ section}} = \begin{bmatrix} 1 & \frac{p_0}{\Delta p} \ell \ln\left(1+\frac{\Delta p}{p_0}\right) & \underline{0} & & \underline{0} \\ 0 & \frac{p_0}{p_0+\Delta p} & & & \\ \underline{0} & & 1 & \frac{p_0}{\Delta p} \ell \ln\left(1+\frac{\Delta p}{p_0}\right) & \underline{0} \\ & & 0 & \frac{p_0}{p_0+\Delta p} & \\ \underline{0} & & \underline{0} & & 1 \end{bmatrix}$$

Important: $\det \underline{M}$ acc section $\neq 1 \Leftrightarrow$ first order derivative term in eqn. of motion!

3.5 Momentum compaction factor:

$$x = C(s)\, x_0 + S(s)\, x_0' + D(s)\, \delta$$

$\Rightarrow$ dispersive trajectories $\Delta x_d = D(s)\,\delta$

$\Rightarrow$ change in path length $\Delta L_d = L_0 - L_d$

from before: $\Delta L_d = \delta \int_0^{L_0} \frac{1}{\rho(s)} D(s)\, ds$

$= \delta\, I_d \propto \delta$

$\Rightarrow$ define momentum compaction factor for circular accelerator:

$$\alpha = \frac{\Delta L_d / L_0}{\Delta p / p_0} = \frac{1}{L_0} \frac{\Delta L_d}{\delta} = \frac{1}{L_0} \oint \frac{1}{\rho(s)} D(s)\, ds$$

L_0: length of ideal, closed orbit in circ. machine

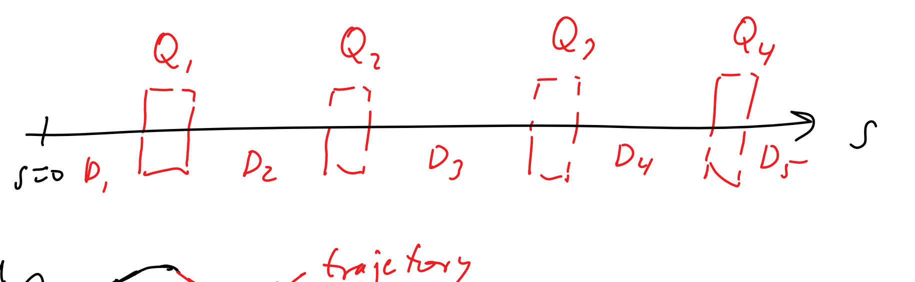
$\Rightarrow \alpha \neq 0 \Rightarrow$ revolution period of particle depends on momentum

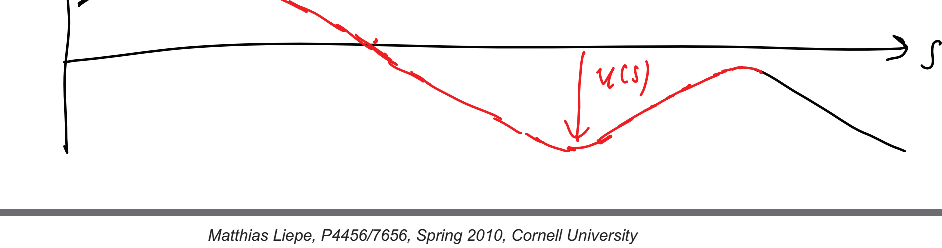
$\Rightarrow$ accelerating phase in RF cavity of particle depends on momentum error

$\Rightarrow$ Longitudinal phase focusing

3.6 Twiss Parameters: α, β, γ, ψ

so far: Matrix formalism $\rightarrow$ calculate individual particle trajectories through arbitrary structures of magnets





in the following: assume $\underline{\delta = 0}$

equation of motion:

$$u'' + K(s)\, u = 0$$

$\Rightarrow$ trajectory: $u(s) = C(s)\, u_0 + S(s)\, u_0'$

for stable orbit: transverse oscillation about the design orbit

= $\underline{\underline{\text{betatron oscillations}}}$

$\Rightarrow$ oscillation amplitude and phase depend on the position s along the orbit

$\Rightarrow$ ansatz: $u(s) = \underbrace{\sqrt{2J\beta(s)}}_{\text{position dependent amplitude of oscil.}} \sin(\psi(s) + \phi_0)$

with J and ϕ_0 defined by initial conditions

$\Rightarrow$ insert into equ. of motion:

$$u'(s) = \sqrt{\frac{2J}{\beta}}\left[\beta\psi' \cos(\psi(s) + \phi_0) - \alpha \sin(\psi(s) + \phi_0)\right]$$

with $\boxed{\alpha(s) \equiv -\frac{1}{2}\beta'(s)}$

note: $\alpha(s)$ is $\underline{\text{not}}$ the momentum compaction factor!

$$u''(s) = \sqrt{\frac{2J}{\beta}}\Big[\left(\beta\psi'' - 2\alpha\psi'\right)\cos\left(\psi(s)+\phi_0\right)$$
$$- \left(\alpha' + \frac{\alpha^2}{\beta} + \beta\psi'^2\right)\sin\left(\psi(s)+\phi_0\right)\Big]$$
$$= -K u = \sqrt{\frac{2J}{\beta}}\left[-K\beta\sin\left(\psi(s)+\phi_0\right)\right]$$

$\Rightarrow$ need:

$$\beta\psi'' - 2\alpha\psi' = 0 \qquad (a)$$
$$\alpha' + \frac{\alpha^2}{\beta} + \beta\psi'^2 = K\beta \qquad (b)$$

$\Rightarrow$ from (a): $\beta\psi'' - 2\alpha\psi' = \beta\psi'' + \beta'\psi' = (\beta\psi')' = 0$

integrate $\Rightarrow$ $\psi' = \frac{I}{\beta}$ $\Rightarrow$ $\psi = \int_0^s \frac{I}{\beta(\tilde{s})}\,d\tilde{s}$

$\Rightarrow$ from (b)

$$\alpha' + \gamma = K\beta \quad \text{with } \gamma = \frac{I^2 + \alpha^2}{\beta}$$

$\Rightarrow$ Universal choice for constant I: $\underline{I = 1}$

$\Rightarrow$ End result: Twiss parameters: $\alpha, \beta, \gamma, \psi$

$$\boxed{\begin{aligned} \beta'(s) &= -2\alpha(s) \\ \alpha'(s) &= K(s)\beta(s) - \gamma(s) \\ \gamma(s) &= \frac{1+\alpha^2}{\beta} \\ \psi(s) &= \int_0^s \frac{1}{\beta(\tilde{s})}\,d\tilde{s} \end{aligned}}$$