Recap: Harmonic Motion

- Simple Harmonic Motion:

$\rightarrow$ $x(t) = X_{max}\cos(\omega t+\phi) \Leftrightarrow v(t) = -V_{max}\sin(\omega t+\phi)$

$V_{max} = \omega X_{max}$

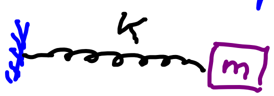
$\Leftrightarrow a(t) = -a_{max}\cos(\omega t+\phi) = -\omega^2 x(t)$

$a_{max} = \omega^2 X_{max} = \omega V_{max}$

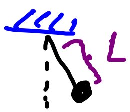
$\rightarrow$ acceleration $a(t) = -\omega^2 x(t) \Rightarrow$ Force $= F_{net} = -m\omega^2 x$

linear restoring force

$\rightarrow$ Mass on spring: $\omega_0 = \sqrt{\frac{k}{m}}$



$\rightarrow$ Simple pendulum: $\omega_0 = \sqrt{\frac{g}{L}}$



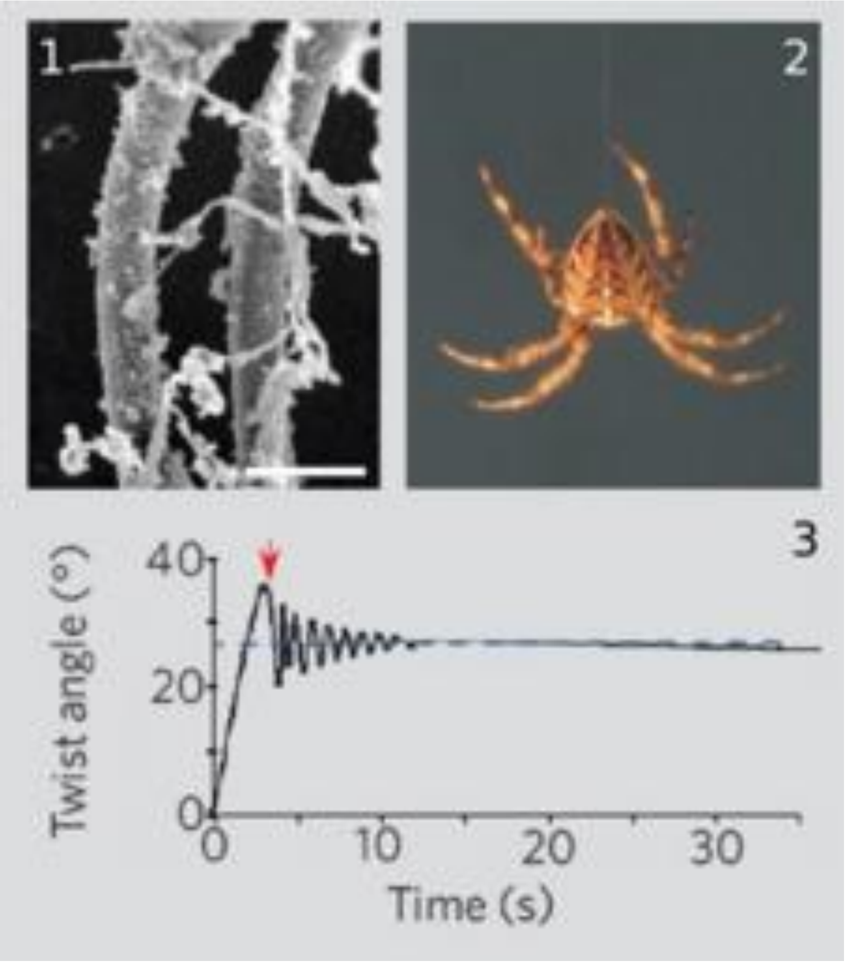
$\omega_0 =$ "natural" angular frequency of oscillation

Today:

- **Damped Oscillations (not damned oscillations…)**
- **Driven Oscillations**

Twist angel of spider suspended from its silk thread

⇒ strong damping (that's why they hardly ever twist!)

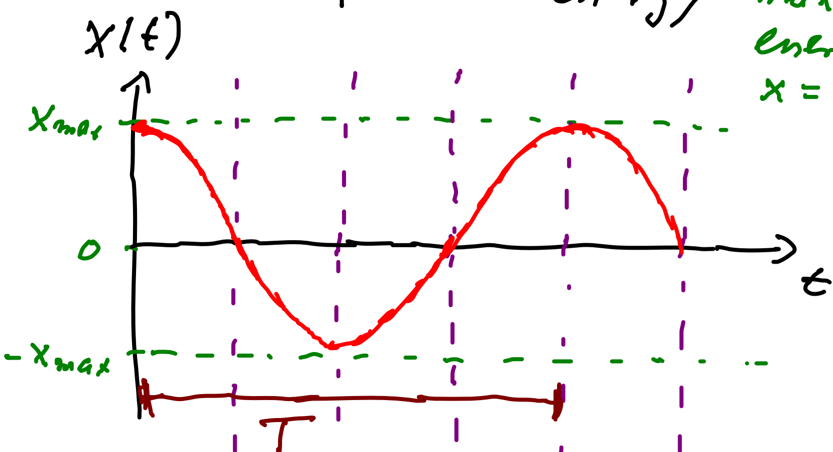


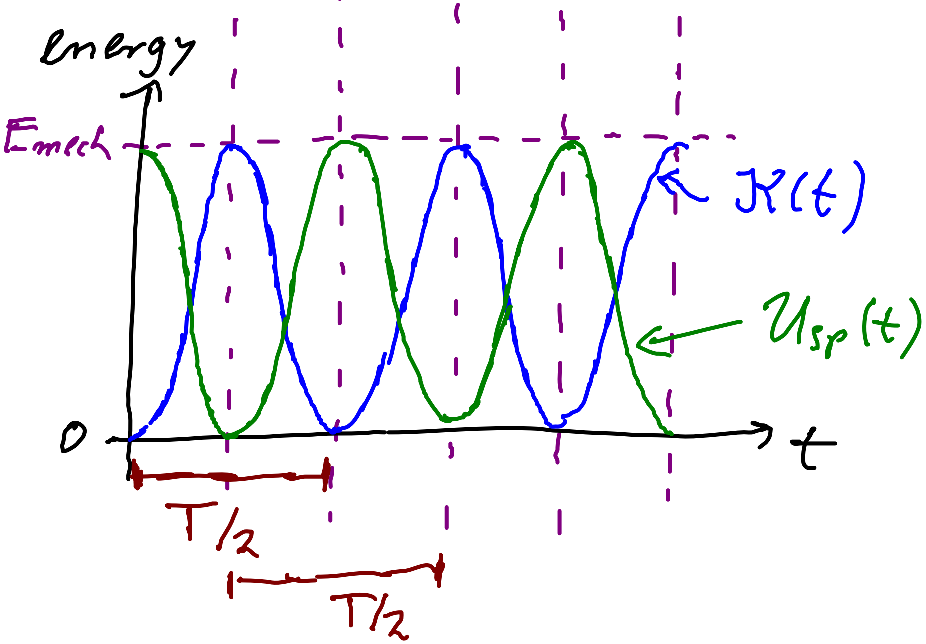
for SHM:

kinetic energy spring const.

$$E_{mech} = K(t) + U_{sp}(t) = \underbrace{\tfrac{1}{2} m v_{max}^2}_{\text{maximum kinetic energy, when } x=0, \text{ and } U=0} = \underbrace{\tfrac{1}{2} k x_{max}^2}_{\text{maximum potential energy, when } x = \pm x_{max} \text{ and } K=0} = const$$

potential energy





Note:

- $K(t)$ and $U(t)$ are 180° out of phase, so that $K(t) + U(t) = const$
- $x(t)$ oscillates at $\omega = 2\pi/T$

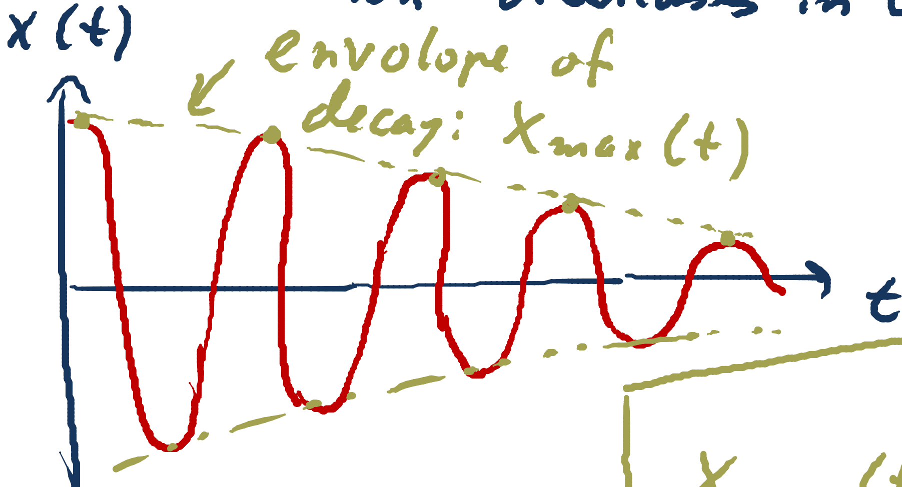
  K, U oscillate at $2\omega = 2\pi/(T/2)$

→ Realistic Case: Damped Oscillations

– with friction $\neq 0$ ⇒ $W_{\text{non-cons. force}} \neq 0$

⇒ E_{mech} decreases in time

⇒ X_{max} decreases in time



$$X(t) = \underline{X_{max}(t)} \cos(\omega t + \phi)$$

in many cases:

$$\boxed{X_{max}(t) = \underbrace{X_{max}(t=0)}_{X_{max,0}}\, e^{-t/(2\tau)}}$$

decays exponentially in time

τ = constant

$[\tau]$ = sec

$\Rightarrow$ for energy:

$$E_{mech}(t) = K(t) + U(t) \approx U_{max}(t) = \frac{1}{2} k\, x_{max}^2(t)$$

$$= \underbrace{\frac{1}{2} k\, x_{max}(t=0)^2}\left(e^{-t/2\tau}\right)^2$$

$$= E_{mech}(t=0) \cdot e^{-t/\tau}$$

$$\Rightarrow \boxed{E_{mech}(t) = E_{mech}(t=0)\, e^{-t/\tau}}$$

τ = energy decay time = time for energy to decay by a factor $1/e = e^{-1} = 0.37$

stronger damping $\Rightarrow$ τ decreases $\Rightarrow$ faster decay

$\Rightarrow$ after time interval $\Delta t = \tau$

$$E_{mech}(t+\tau) = E_{mech}(t=0)\, e^{-(t+\tau)/\tau}$$

$$= \underbrace{E_{mech}(t=0)\, e^{-t/\tau}} \cdot \underbrace{e^{-\tau/\tau}}$$

$$= E_{mech}(t) \cdot \frac{1}{e} = 0.37 \cdot E_{mech}(t)$$

$\Rightarrow$ in each time interval $\Delta t = \tau$, the mechanical energy decays by a multiplicative factor $1/e = 0.37$.

$t = 0$ $\qquad$ $t = \tau$ $\qquad$ $t = 2\tau$

$$E = E_0 \xrightarrow{\Delta t = \tau} E(\tau) = \frac{1}{e} E_0 \xrightarrow{\Delta t = \tau} E(2\tau) = \frac{1}{e}\left(\frac{1}{e}E_0\right) = \frac{E_0}{e^2}$$

$$X_{max} = X_{max,0} \qquad X_{max}(\tau) = \frac{X_{max,0}}{\sqrt{e}} \qquad X_{max}(2\tau) = \frac{1}{\sqrt{e}}\left(\frac{X_{max,0}}{\sqrt{e}}\right) = \frac{X_{max,0}}{e}$$

$$X_{max}(t) = X_{max,0}\, e^{-t/2\tau} \qquad e^{-1/2} = \frac{1}{\sqrt{e}}$$

equal time intervals $\Rightarrow$ equal multiplication factors in E_{max}, X_{max}, V_{max} ....

similar:

$t = 0$ $\qquad$ $t = 10\,s$ $\qquad$ $t = 20\,s = 2 \cdot 10\,s$

$$E_0 \xrightarrow{\Delta t = 10\,s} E(10\,s) = \frac{1}{c} E_0 \xrightarrow{\Delta t = 10\,s} E(20\,s) = \frac{1}{c}\left(\frac{E_0}{c}\right) = \frac{E_0}{c^2}$$

same factor $\frac{1}{c}$ every $\Delta t = 10\,s$

The mechanical energy of a damped harmonic oscillator decays by a **factor of 4 in 10 s**.

$E_0 \rightarrow \frac{1}{4} E_0$ in $\Delta t = 10s$

How long does it take to decay by a **factor of 16**?

exponential decay:

$E_{mec}(t=10s) = E_{mec}(t=0)\, e^{-10s/\tau} = \frac{1}{4} E(t=0)$

$t=0$ $\quad$ $t=10s$ $\quad$ $t=20s$

$E(0) \rightarrow E = \frac{E_0}{4} \longrightarrow E = \frac{1}{4}\left(\frac{E_0}{4}\right) = \frac{E_0}{16}$

$E_{mec}(t=20s) = E(t=0)\, e^{-20s/\tau}$

$= E(0)\, \underbrace{e^{-10s/\tau}}_{1/4} \cdot \underbrace{e^{-10s/\tau}}_{1/4}$

A. 10 s

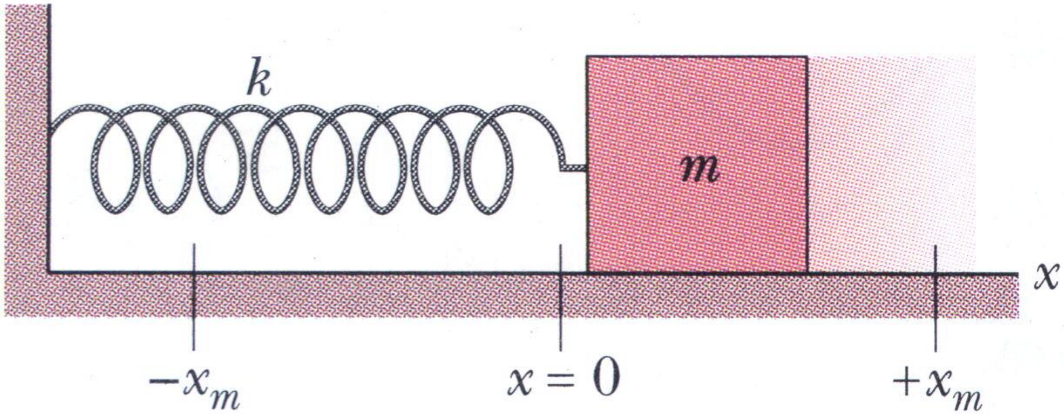
B. 20 s

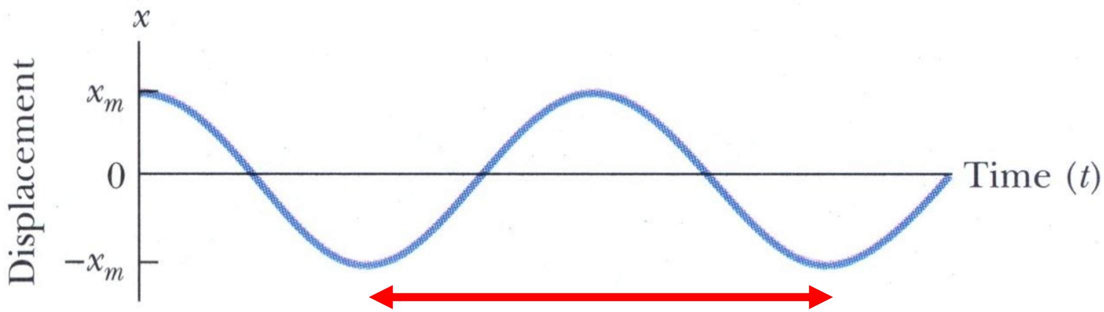
C. 40 s

D. 80 s

E. 160 s

Harmonic Oscillator:





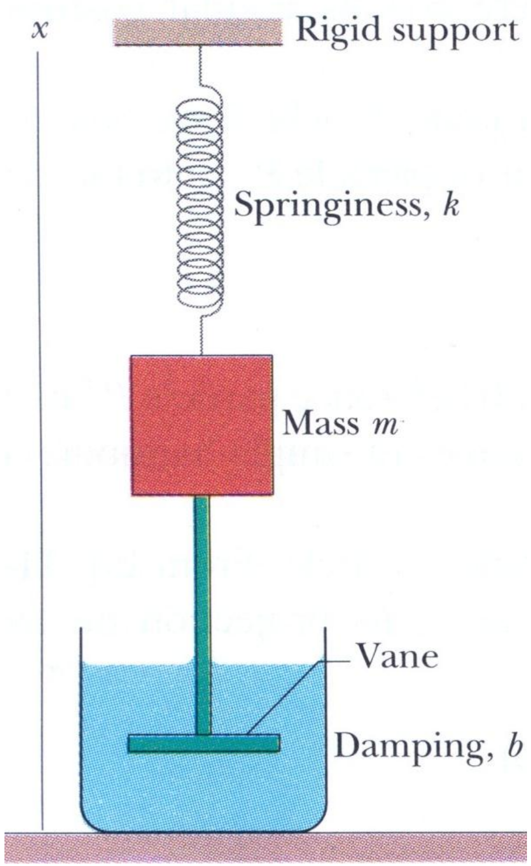
$T = 2\pi / \omega_o$

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

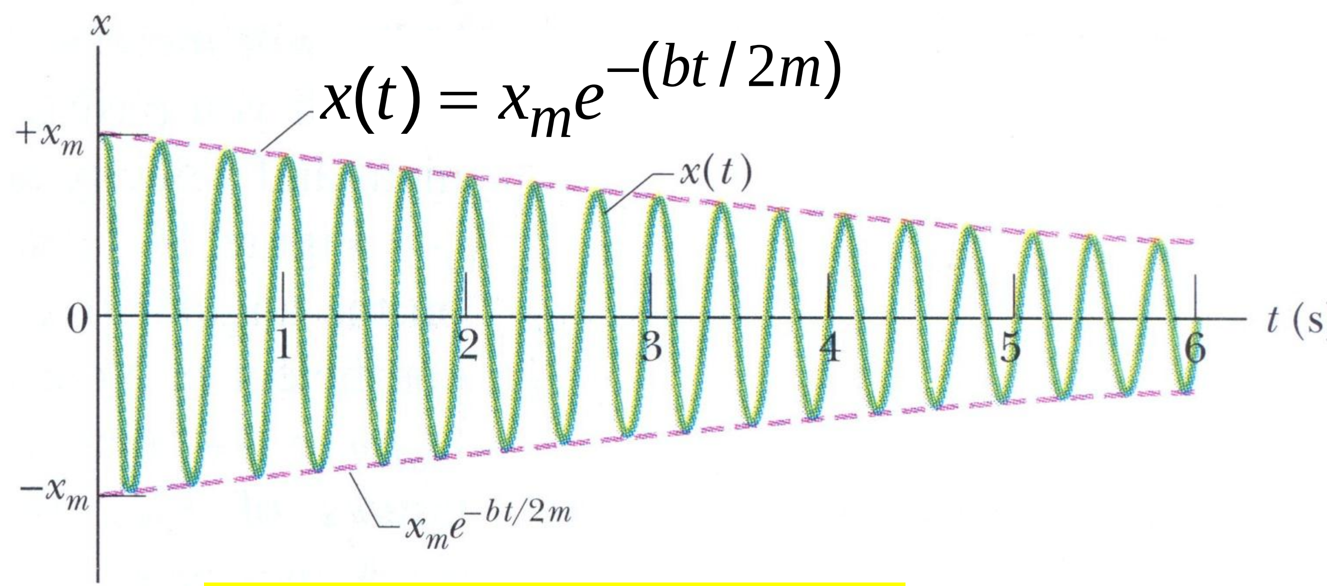
natural
angular frequ.
without friction

Damped Harmonic Oscillator:

Damping force: $F = -bv$



$$x(t) = x_m e^{-(bt/2m)}$$



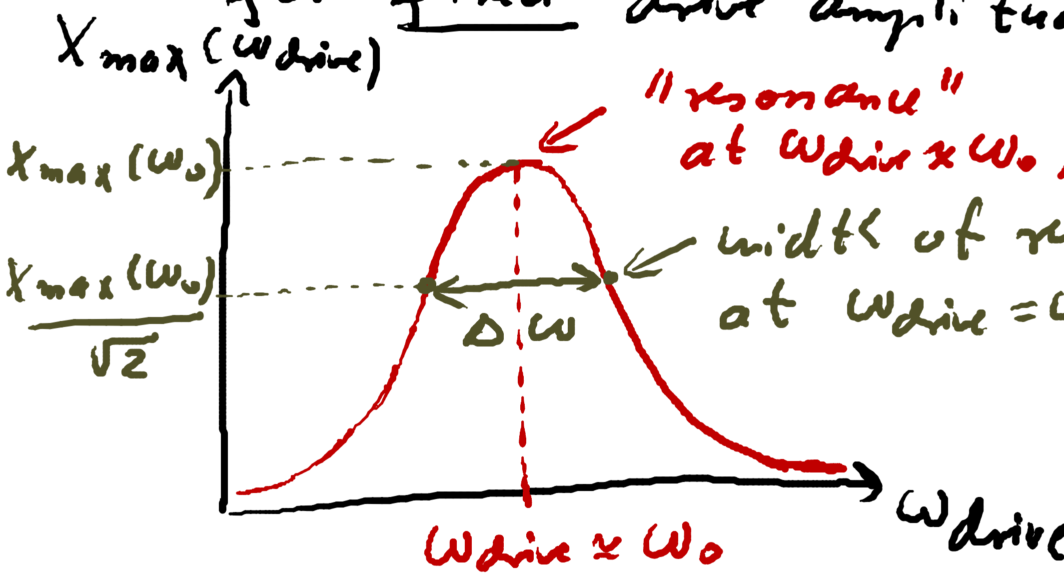
$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \approx \sqrt{\frac{k}{m}} = \omega_0$$

Energy decay time $\tau = m/b$

- Forced (driven) oscillations and resonance:
  - drive a damped oscillator continuously at frequency ω_{Drive} = drive frequency

$$\boxed{\text{Response (motion) will be SHM at } \omega_{drive} \text{ and not } \omega_0!}$$

  - measure response amplitude X_{max} vs. ω_{drive} for fixed drive amplitude:



"resonance" at $\omega_{drive} \approx \omega_0$, i.e. at natural frequency

width of resonance peak: $\Delta\omega$ at $\omega_{drive} = \omega_0 \pm \frac{\Delta\omega}{2}$:

$$X_{max} = \frac{X_{max}(\omega_0)}{\sqrt{2}}$$

$$E_{mech} = \frac{E_{mech}(\omega_0)}{2} \leftarrow E_{mech} = U_{max} \propto X^2$$

for many systems:

- resonance width $\boxed{\Delta\omega = \frac{1}{\tau}}$ τ energy decay time

- define quality factor: $\boxed{Q \equiv \frac{\omega_0}{\Delta\omega} = \omega_0\tau = \frac{2\pi\tau}{T_0}}$

$= 2\pi\left(\begin{array}{l}\#\text{ of oscillations at}\\ \omega_0 \text{ in time interval } \Delta t = \tau\end{array}\right)$

- amplitude at resonance $\propto Q \propto \tau$

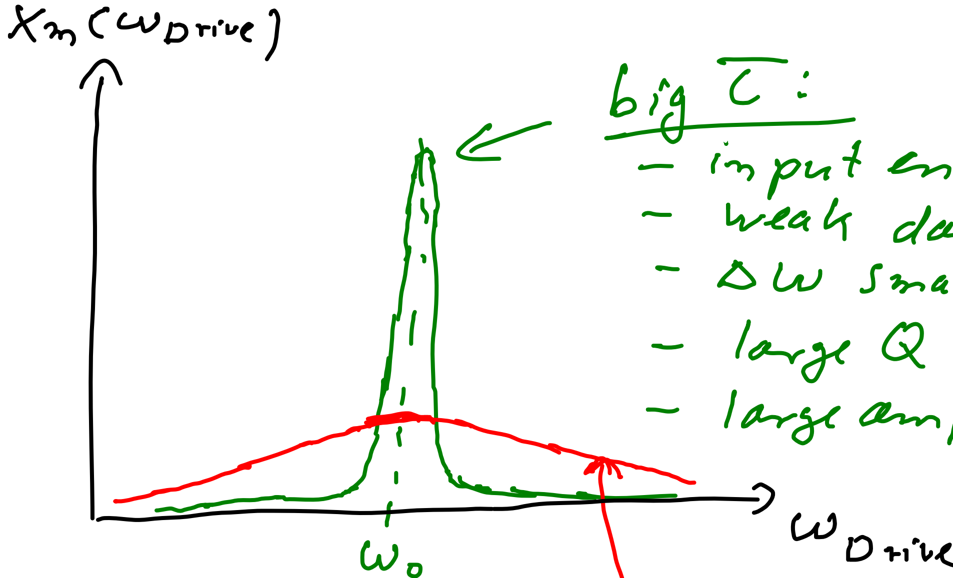
$\propto$ "time the system remembers energy input"

- specific model:

$$X_{max}(\omega_{drive}) = \frac{\overset{\text{force amplitude}}{F_{max}}/\underset{\text{mass of obj.}}{m}}{\sqrt{(\omega_{drive}^2 - \omega_0^2)^2 + \left(\frac{\omega_{drive}}{\tau}\right)^2}}$$

$\Rightarrow$ max. amplitude at

$\omega_{drive} = \sqrt{\omega_0^2 - \frac{1}{2\tau^2}} \approx \omega_0$ for small damping, i.e. large τ



big τ:

- input energy decays slowly
- weak damping
- $\Delta\omega$ small
- large Q
- large amplitude at resonance

small τ:

- input energy decays fast
- strong damping
- $\Delta\omega$ large
- small Q
- small amplitude at resonance

Resonance and Driven Oscillations in Animal Movement

Problem: Nature didn't invent the wheel

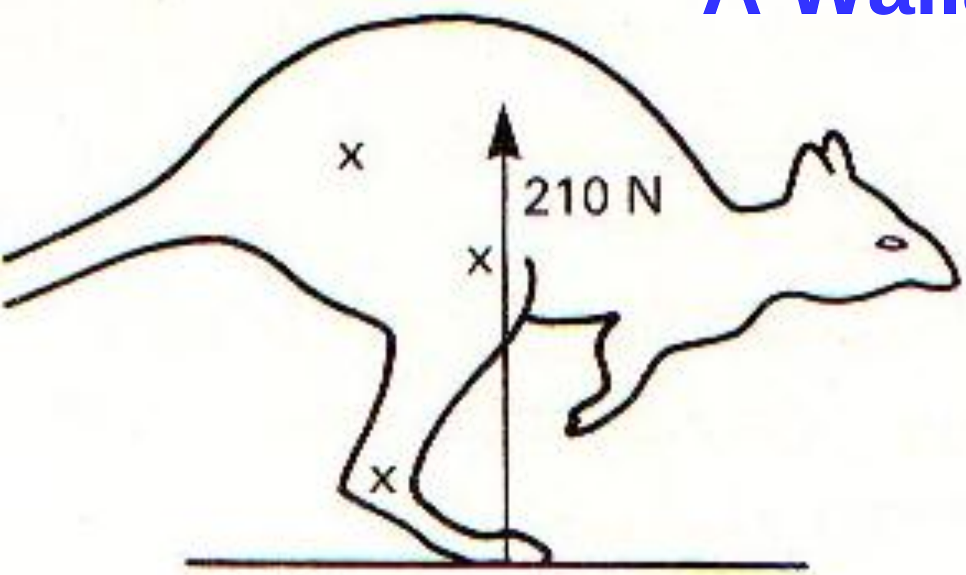
- Animal motion requires that parts of the animal speed up and slow down, with a corresponding variation in kinetic energy.
- Doing this with muscles alone would waste too much energy.

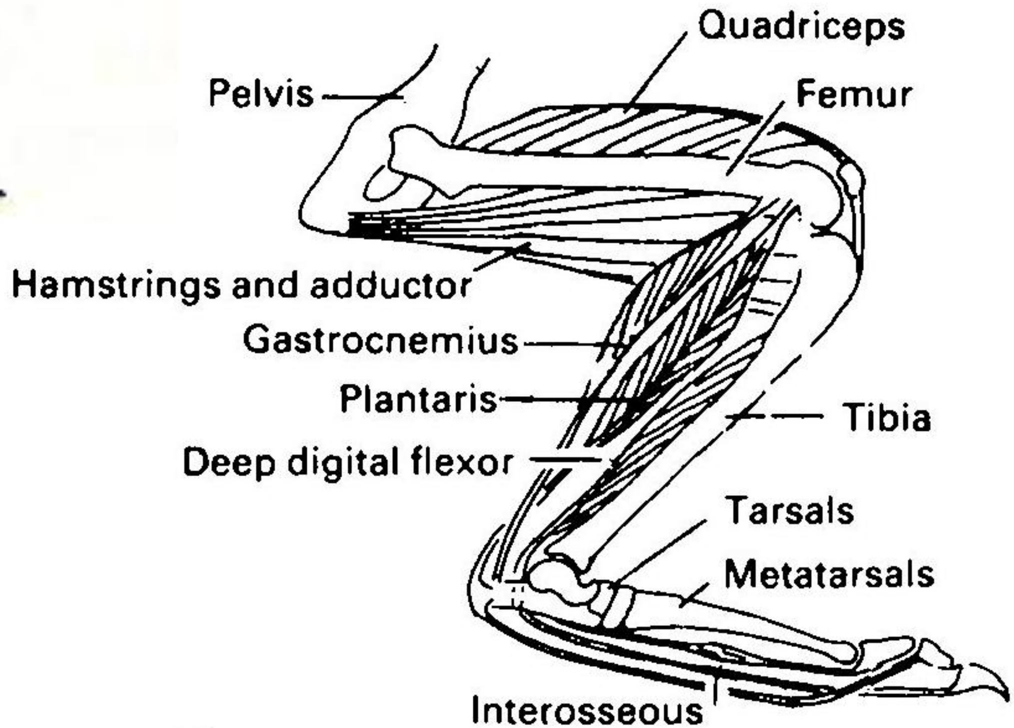
Solution:

- Use **springs** (**tendons, ligaments, muscles**) to convert **kinetic energy** to and from **elastic potential energy**.

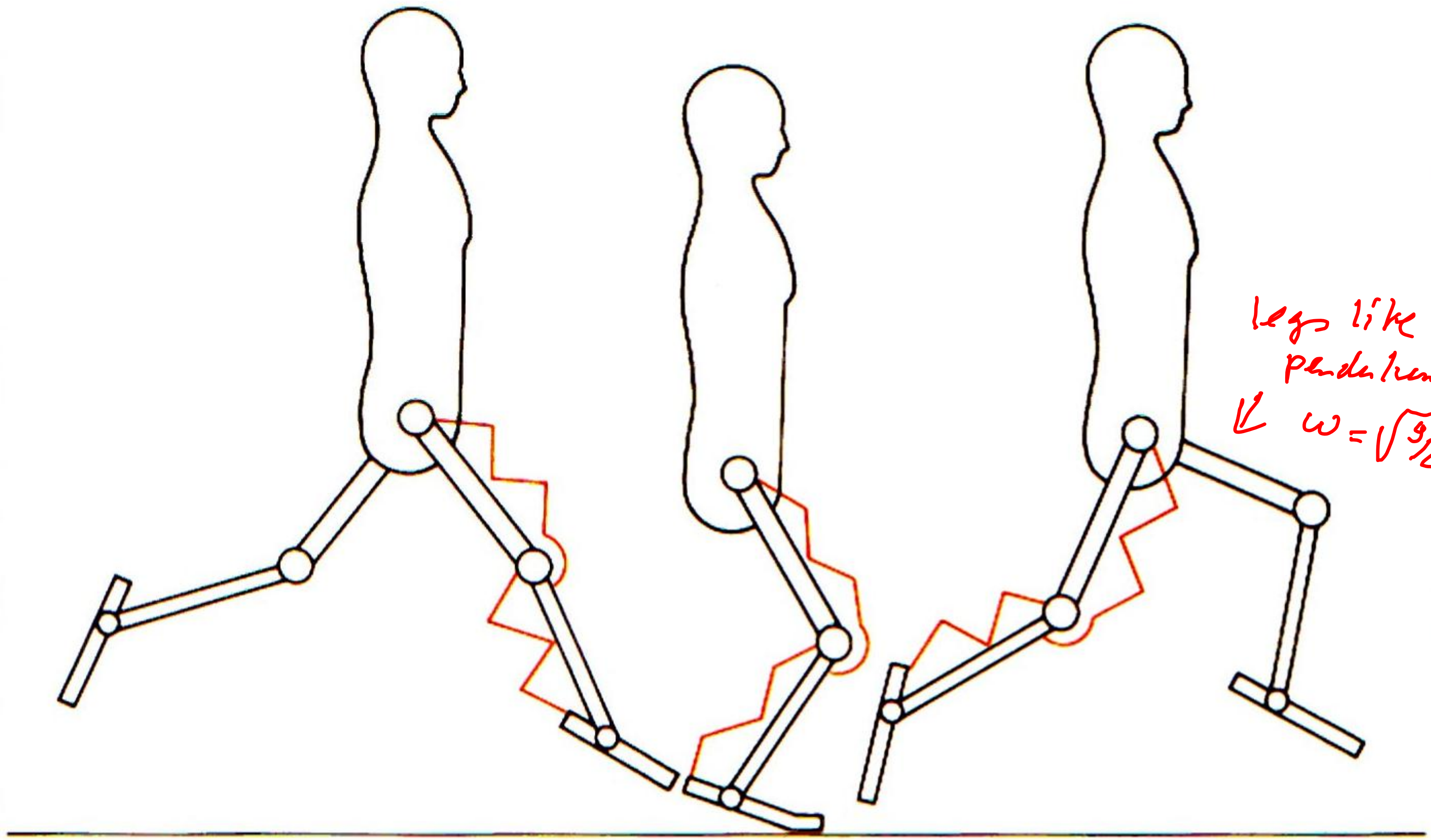
- **Drive the system near its resonant frequency to get the maximum motion for a given energy input.**

A Wallaby Hopping:





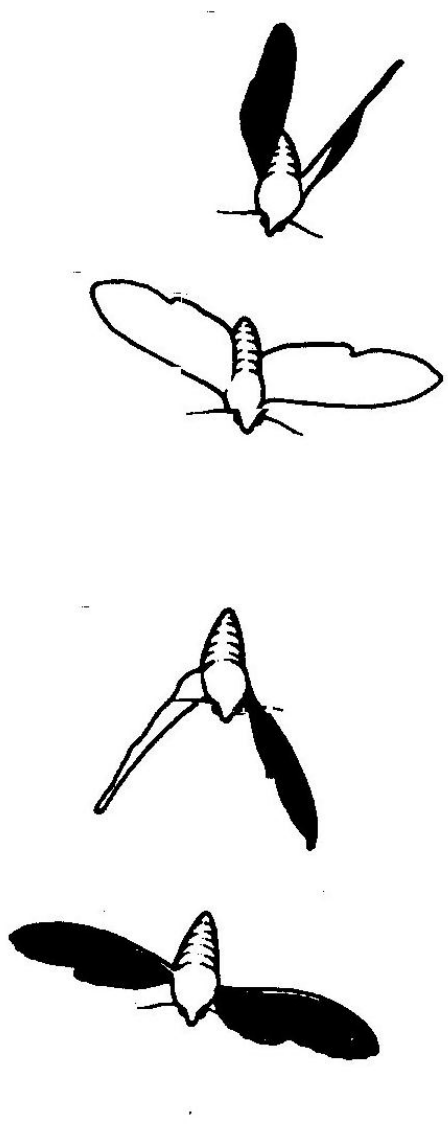
A Human Running:

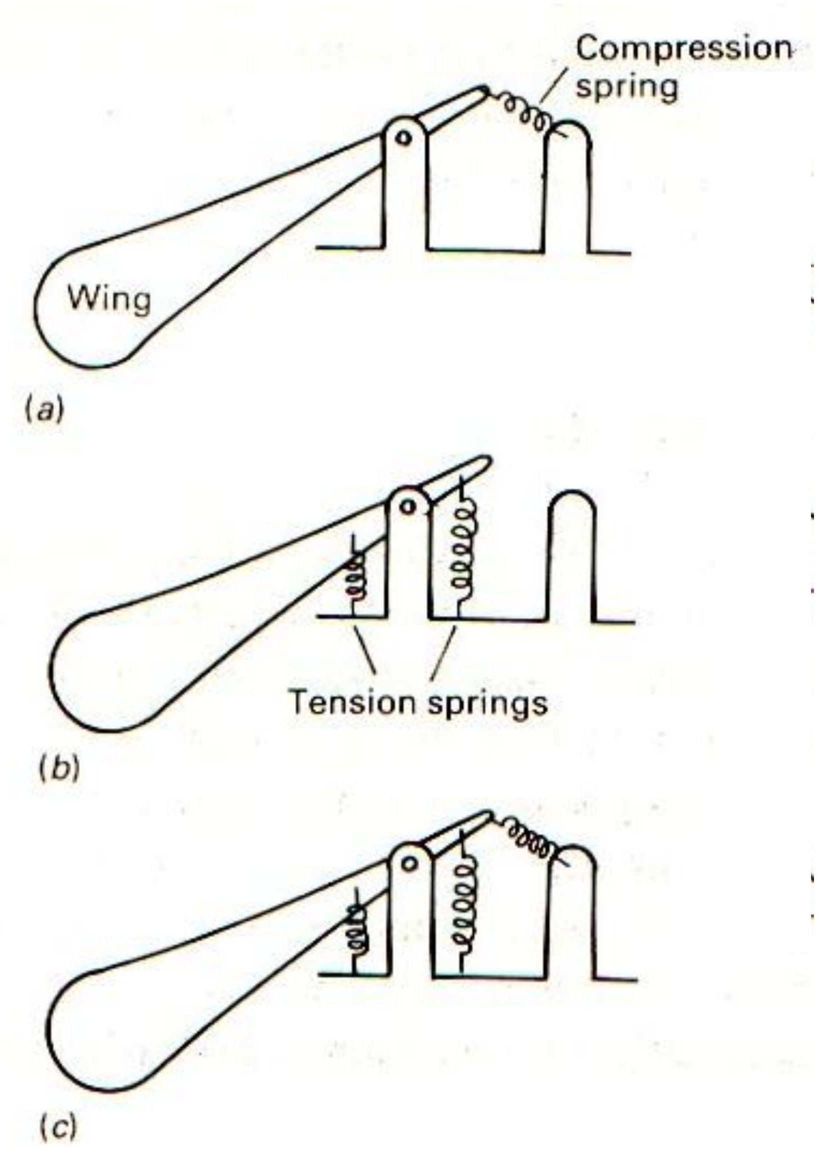


legs like pendulum

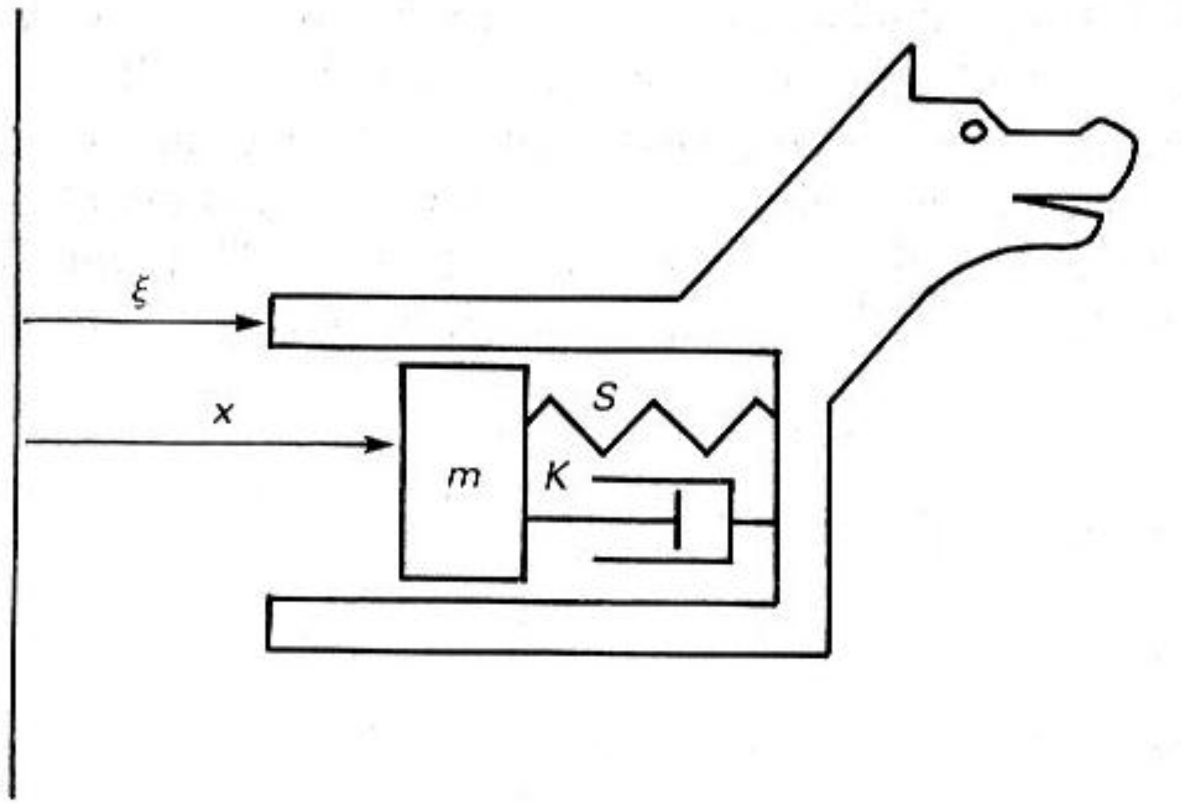
$\omega = \sqrt{g/L}$

An Insect Flying:





A Dog Panting:



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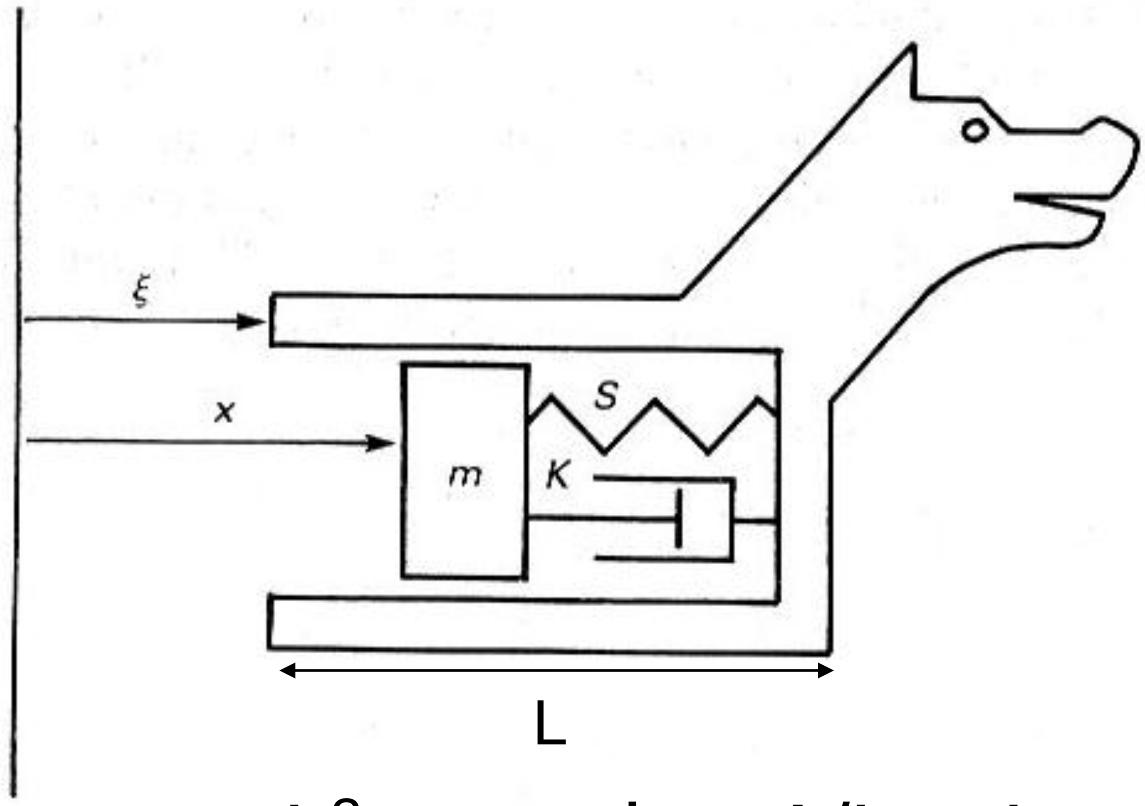

$m \propto L^3, \qquad k \propto A/L \propto L$

$\therefore \quad T = 2\pi(m/k)^{1/2} \quad \Rightarrow \quad \mathbf{T \propto L \propto}$ **size of dog**