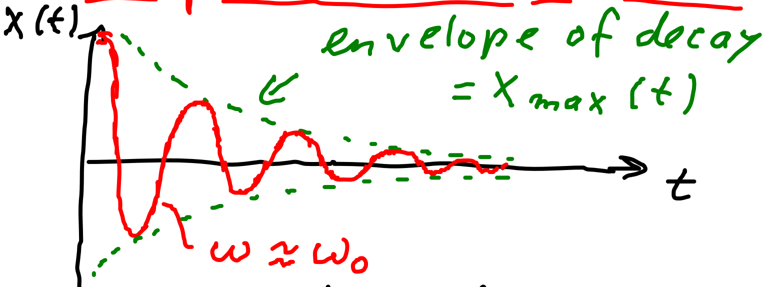
Recap: Harmonic Motion

Lecture 21

• Damped Harmonic Motion:



$$x(t) = x_{max}(t)\cos(\omega t + \phi)$$

$$x_{max}(t) = x_{max}(t=0)\, e^{-t/(2\tau)}$$

note "2" here!

$$\Rightarrow E_{mech}(t) = E_{mech}(t=0)\, e^{-t/\tau}$$

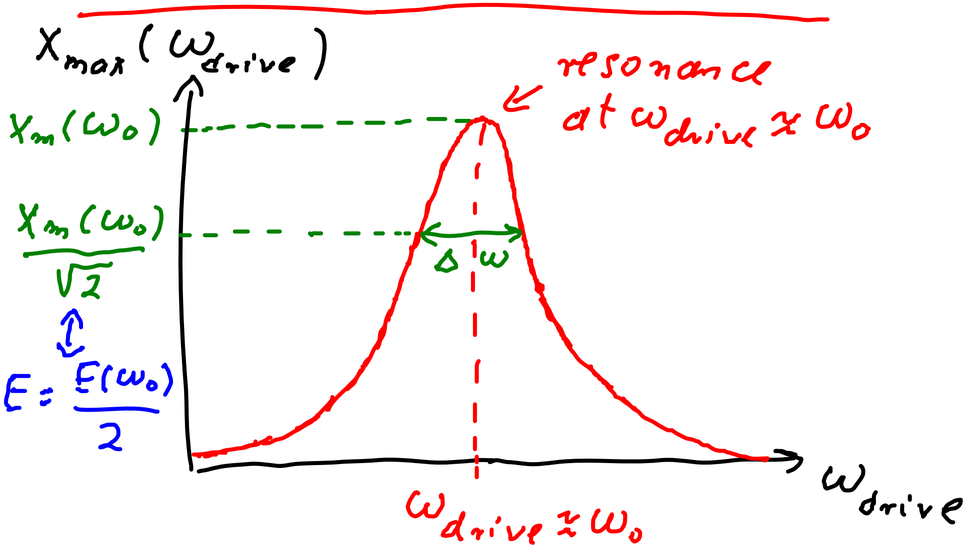
τ: energy decay time

math: $(e^a)^b = e^{ab}$

$e^{a+b} = e^a \cdot e^b$

in each $\Delta t = \tau$: $\begin{cases} x_{max} \longrightarrow x_{max}/\sqrt{e} \\ E_{mech} \longrightarrow E_{mech}/e \end{cases}$

• Forced (driven) oscillations:



strong damping

$\Rightarrow$ short τ

$\Rightarrow$ large $\Delta\omega = \frac{1}{\tau}$

$\Rightarrow$ small quality factor $Q \equiv \omega_0 \tau$

$\Rightarrow$ small amplitude at resonance $\propto Q$

What is the initial speed you would have to give an object so that it can escape from the Earth's surface and never come back, i.e. go infinite far away from Earth (**escape speed**)?

v = ?

A. 25 mi/h

B. 250 mi/h

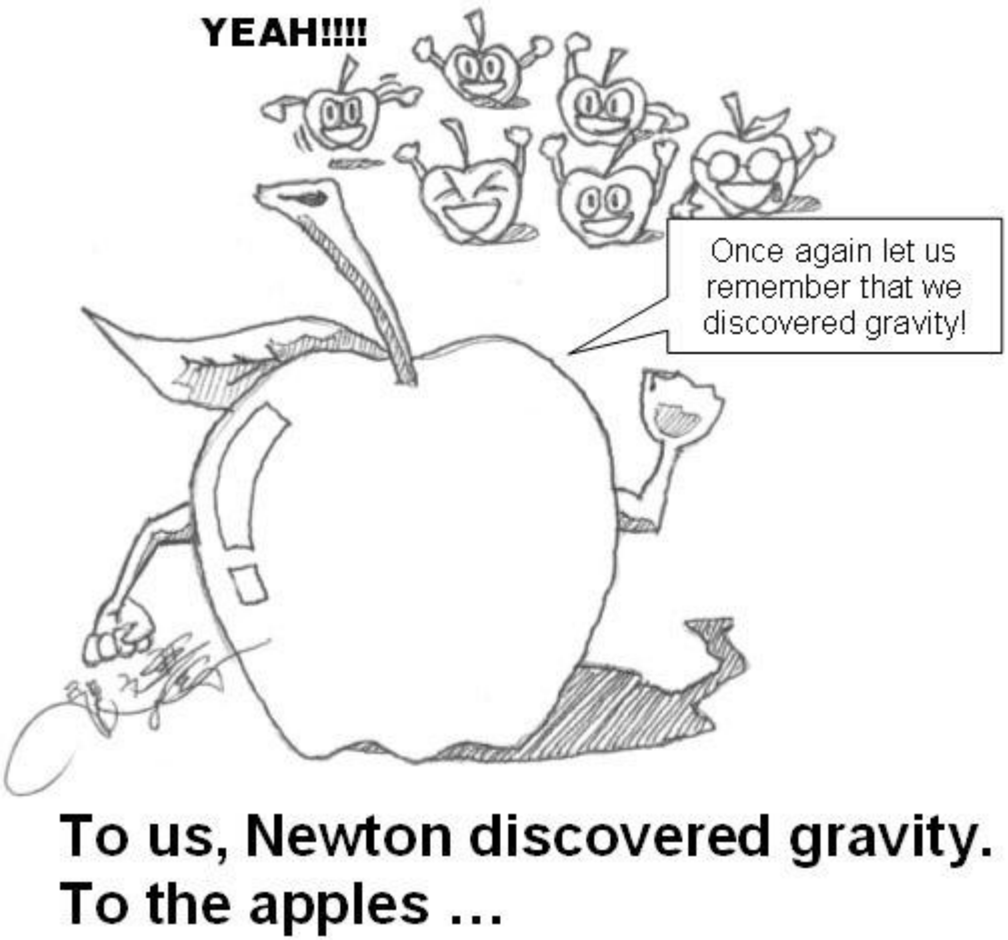
C. 2,500 mi/h

D. 25,000 mi/h ≃ 7 mi/sec

E. 250,000 mi/h

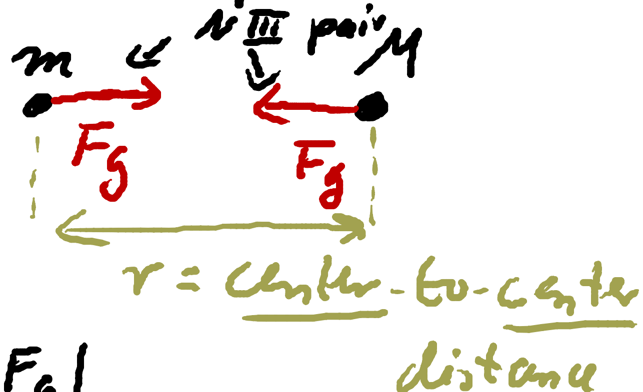
Today:

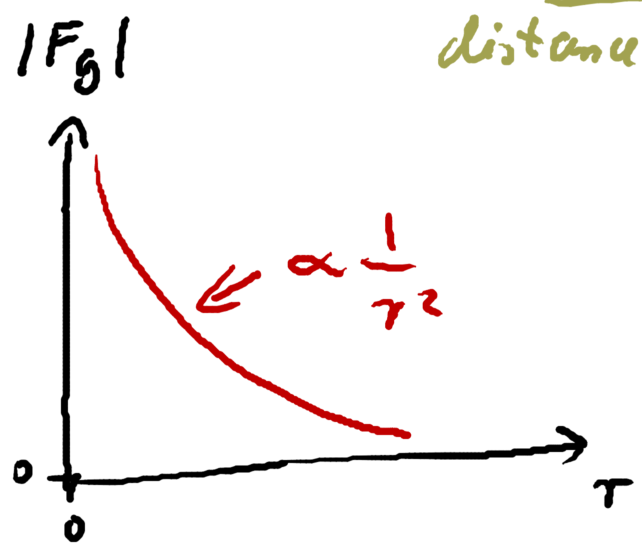
- **Newton's law of gravitation**
- **Satellite motion**
- **Energy of orbital motion**
- **Escape speed**



Gravity:

F_g : gravitational force between two objects; "action at a distance" force; **always** attracts objects





Newton's Law of Gravitation

$$\boxed{|F_g(r)| = G\,\frac{mM}{r^2}} \propto \frac{1}{r^2}$$

$$G = 6.67\cdot 10^{-11}\,\frac{Nm^2}{kg^2}$$

: universal gravitation constant

__Note I:__ $F_g = m \underbrace{g}_{10\,m/s^2}$ } for grav. force by earth on object on Earth's surface <u>only</u>

<u>near Earth's surface:</u>

$$F_g(\underset{\substack{\uparrow\\ \text{earth}}}{r = r_E}) = G\frac{m M_E}{r_E^2} = m\underbrace{\left(\frac{G M_E}{r_E^2}\right)}_{\substack{= 10\,m/s^2 \\ = \text{acceleration from Earth's gravity at Earth's surface}}} = mg$$

$\Rightarrow$ more generally:

$$\left(\begin{array}{c}\text{acceleration of an}\\ \text{object at }\underline{\text{radial}}\\ \text{distance } r \text{ from}\\ \text{the center of the}\\ \text{earth by gravity}\end{array}\right) = g(r) = \frac{G M_E}{r^2}, \quad r > r_E$$

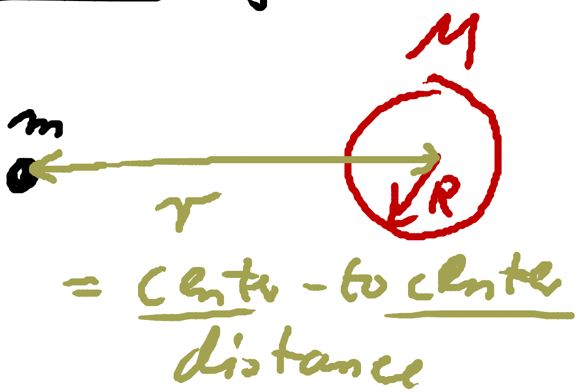
Note II:

$F_g = 6.67 \cdot 10^{-11}\,N = 67\,pN$

$\Rightarrow$ very weak force

$\Rightarrow$ only matters when at least one of the objects mass is large

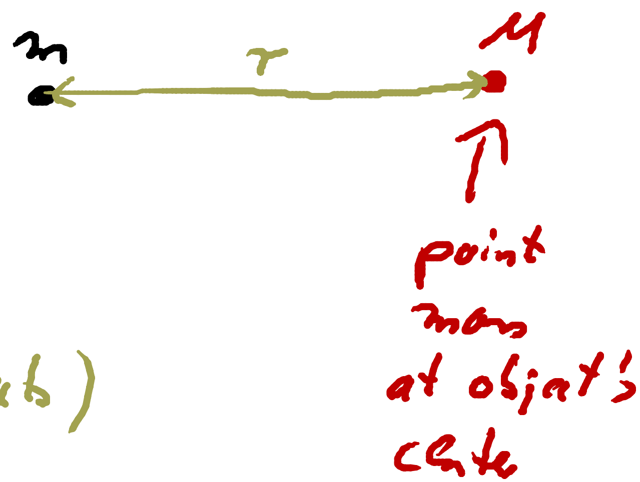
Note III: for distributed mass (planets ...)



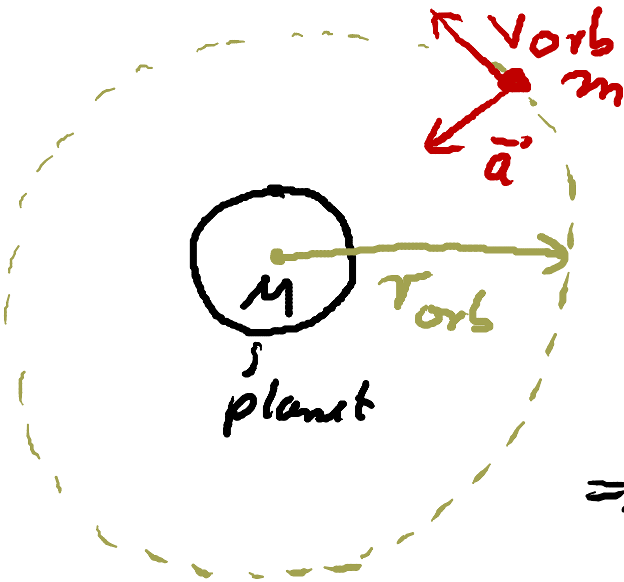
$\equiv$

for $r > R$

(and uniform, spherical objects)

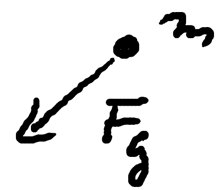


→ Satellite Motion



"satellite" in uniform circular motion on orbit

FBD:



$N\,II$: $\Sigma F = F_g = ma$

$$\Rightarrow F_{\substack{\text{by gravity}\\ \text{on } m}} = G\frac{mM}{r_{orb}^2} = ma = m\frac{v_{orb}^2}{r_{orb}}$$

$$\Rightarrow v_{orb} = \sqrt{\frac{GM}{r_{orb}}}$$

orbital speed of satellite, depends on M, but <u>not</u> on mass m of satellite!

$r_{orb} \to 2r_{orb}$

If you double the orbital radius of a satellite in a circular orbit, what must happen to its orbital velocity?

$$V_{orb} = \sqrt{\frac{GM}{r_{orb}}}$$

$$\Rightarrow V_{orb} \propto \frac{1}{\sqrt{r}}$$

$$\Rightarrow \frac{V_{orb}(2r)}{V_{orb}(r)} = \frac{\sqrt{r}}{\sqrt{2r}} = \frac{1}{\sqrt{2}}$$

v_f = ?

A. v_i

B. $\sqrt{2}\, v_i$

C. $2\, v_i$

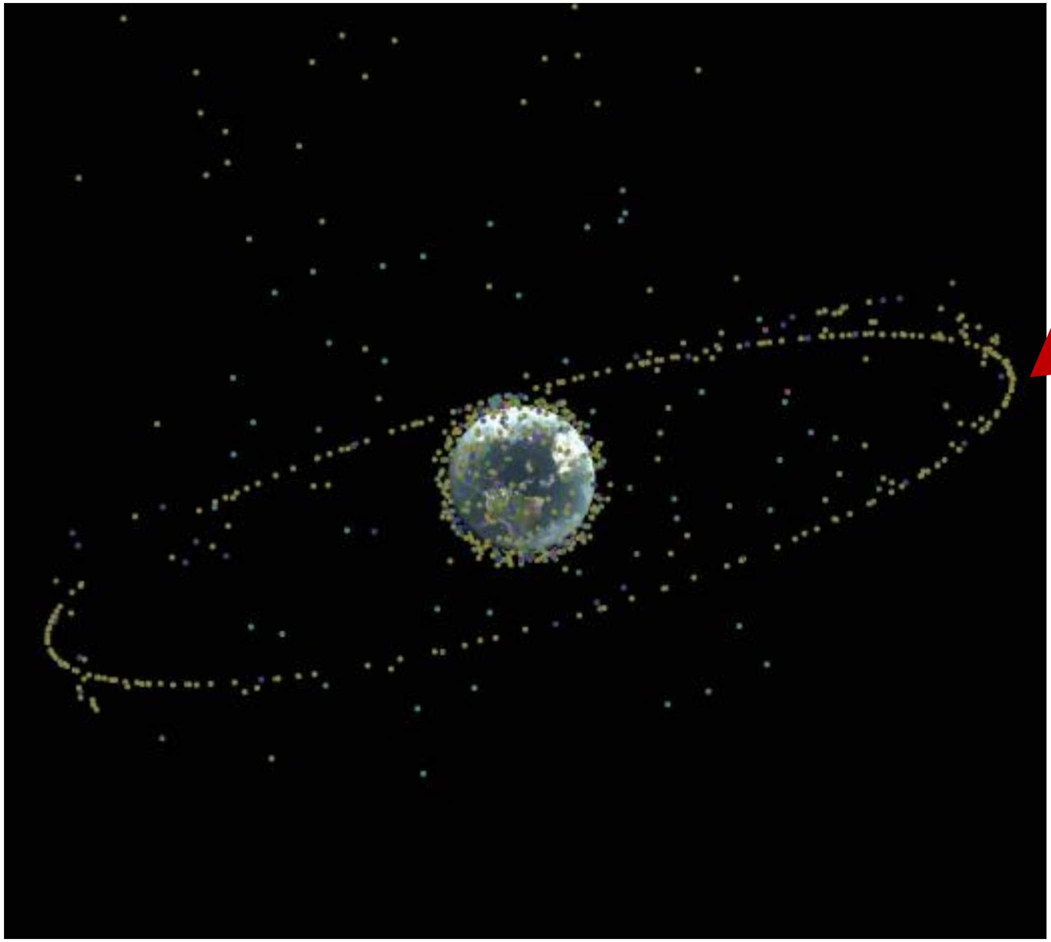
D. $1/\sqrt{2}\, v_i$

E. $1/2\, v_i$

$\Rightarrow$ periode:

$$T = \frac{2\pi r_{orb}}{V_{orb}} \propto r_{orb} \cdot \sqrt{r_{orb}} = r_{orb}^{3/2}$$

Satellites orbiting Earth



Geosynchronous orbit: T = 1 day

Energy of satellite in Orbit (orbital motion)

• Gravitational potential energy: $U_g(r)$

$F_g \leftarrow m \bullet \longrightarrow +r$

choose $\boxed{U_g(r=\infty)=0}$

move satellite from ∞ to r_{orb}

$$\Delta U_g = U_g(r) - \underbrace{U_g(\infty)}_{=0} = U_g(r) = -W_g$$

$$= -\int_\infty^r \underbrace{F_g(r)}_{F_g \text{ points in } -r \text{ direction}} dr = +\int_\infty^r G\frac{mM}{r^2}\,dr = -G\frac{Mm}{r}$$

$$\Rightarrow \boxed{U_g(r) = -G\frac{mM}{r}} \le 0 \qquad F_g = -G\frac{mM}{r^2}$$

not $1/r^2$ ← check: $F_g(r) = -\frac{dU_g(r)}{dr}$ ✓

- use for large changes in r, or if not on Earth's surface
- use $\Delta U_g = mg\Delta y$ for small changes Δy near Earth's surface only

- Kinetic Energy $K_{orb}(r)$

$$K_{orb}(r_{orb}) = \frac{1}{2} m v_{orb}^2 = \frac{1}{2} m G \frac{M}{r_{orb}} = -\frac{1}{2} U_g(r_{orb})$$

$$v_{orb} = \sqrt{\frac{GM}{r_{orb}}}$$

≥ 0 always

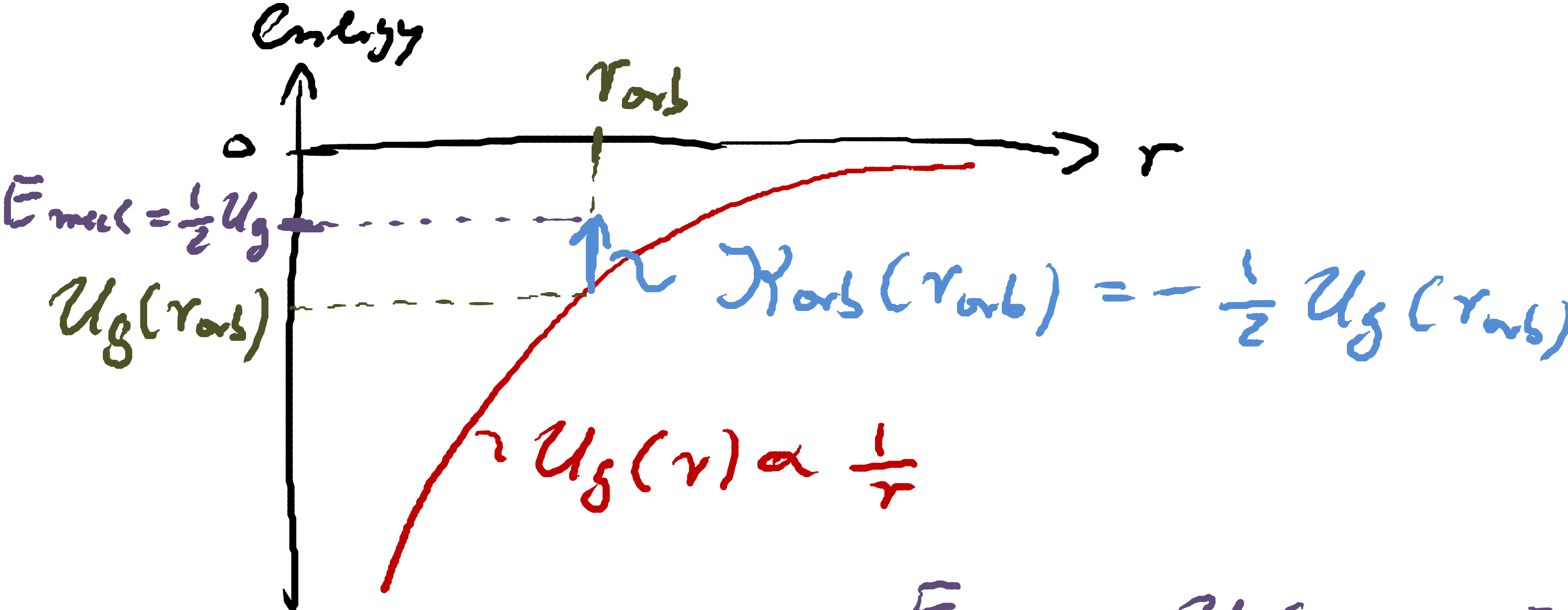
(note $K \geq 0$ always)

- Mechanical Energy:

$$E_{mech,\,orb}(r_{orb}) = K_{orb}(r_{orb}) + U_g(r_{orb})$$

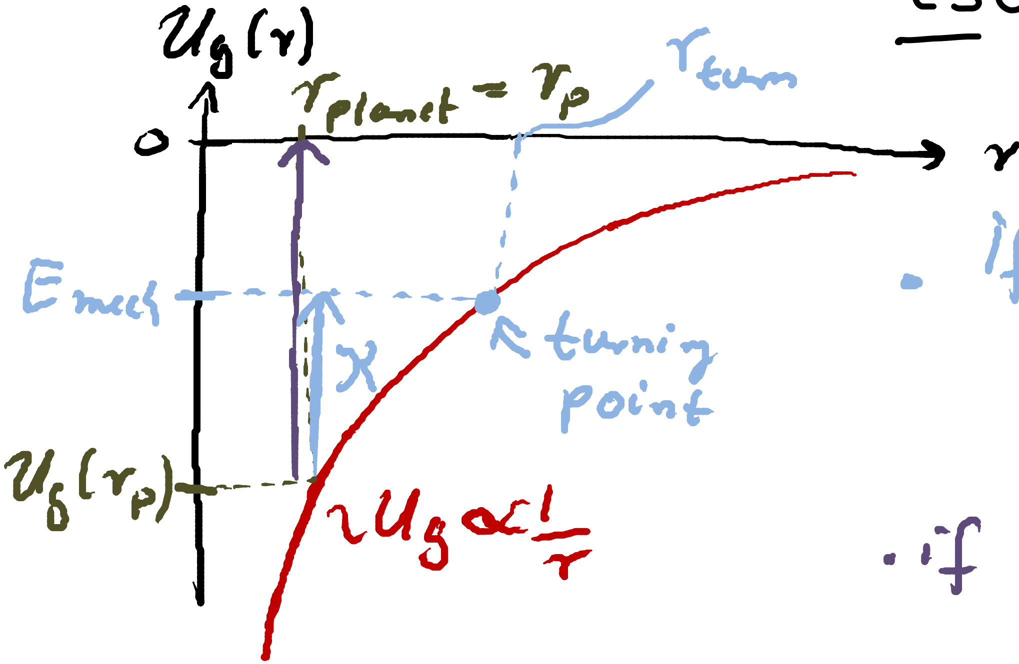
$$= -\frac{1}{2} U_g(r_{orb}) + U_g(r_{orb}) = \frac{1}{2} U_g(r_{orb})$$

$$= -\frac{1}{2} G \frac{mM}{r_{orb}} \leq 0$$



$$E_{mech} = U_g(r_{orb}) + K(r_{orb})$$
$$= \frac{1}{2} U_g(r_{orb})$$

Escape Speed:



- $E_{mech} = K + U_g(r)$
- if $E_{mech} < 0 \Rightarrow$ bound state
  $\Rightarrow$ turning point ($K = 0$)
  $E_{mech} = U_g(r = r_{turn})$
- if $E_{mech} \geq 0 \Rightarrow$ no bound state
  $\Rightarrow$ object can reach $r \to \infty$

$\Rightarrow$ minimum kinetic energy that we must give to an object at the surface of the planet, so that it can escape to $r \to \infty$

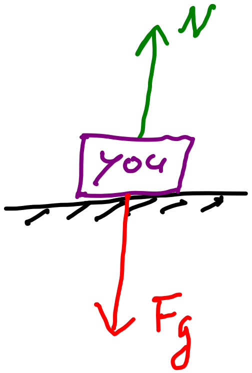
$$K_{esc} = -U_g(r_{planet}) \Leftrightarrow \text{gives } E_{mech} = 0$$

$$\Rightarrow \frac{1}{2} m v_{esc}^2 = G \frac{mM}{r_p}$$

escape speed : minimum speed at surface of planet to go from $r = r_p$ to $r = \infty$

$$\Rightarrow \boxed{v_{esc} = \sqrt{\frac{2GM}{r_p}}}$$

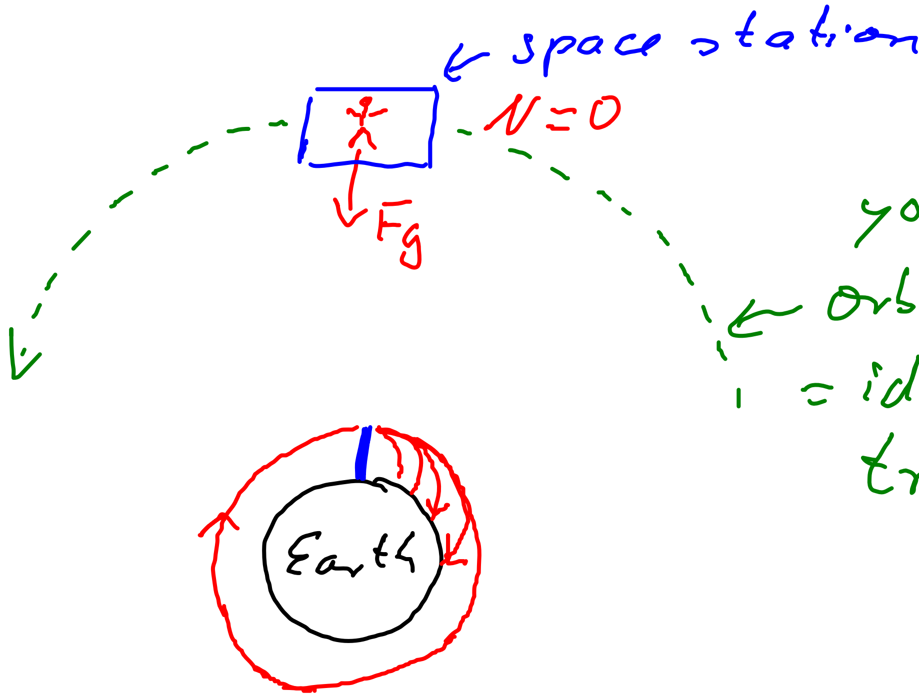
Apparent Weight:



apparent weight
= sensation of weight because we are squished between F_g and N

$= |N|$

$\Rightarrow$ apparent weightlessness, if $N = 0$, and only force acting on you is F_g
(e.g. free fall, astronaut in space station, ...)



you in orbital motion = ideal projectile motion trajectory

$\Rightarrow$ true weightlessness : $F_g = 0$