Cornell University

Department of Physics

Phys 651 September 27, 2006

Relativistic Quantum Field Theory, Fall 2006

Homework Assignment # 5

(Due Wednesday, October 4, after the lecture.)

Lectures and Reading Assignments:

Readings are from *"An Introduction to QFT"* by Peskin and Schroeder.

- Lec 11, 10/02 (Mon): Discrete Symmetries of the Dirac Theory. **Sec. 3.6.**
- Lec 12, 10/04 (Wed): Interacting Fields and Perturbation Theory **Sec. 4.1.**

Problems:

1. **Energy-Momentum Tensor of Dirac Fermions:**

   (a) Show that the Dirac lagrangian is invariant (up to an addition of a total derivative term, $\partial_\mu \mathcal{J}^\mu$) under space-time translations, $x^\mu \to x^\mu + a^\mu$.

   (b) Derive the energy-momentum tensor of the Dirac field – the Noether current corresponding to the symmetry you have just proven.

   (c) Construct the expression for the conserved physical momentum $\mathbf{P}$ in the quantum theory (that is, express it in terms of raising and lowering operators, $a^s_{\mathbf{p}}, b^s_{\mathbf{p}}$, etc.).

   (d) Show that the one-particle states defined in class are indeed states of definite momentum.

2. **Particles and Antiparticles:** In Problem Set 3, pr. 3, you proved that the transformation $\Psi(x) \to e^{i\alpha}\Psi(x)$ is a symmetry of the Dirac lagrangian, and derived the corresponding Noether current, $j^\mu = \bar{\Psi}\gamma^\mu\Psi$. The corresponding conserved charge is given by

$$Q = \int d^3x \Psi^\dagger(x)\Psi(x). \tag{1}$$

   Obtain the expression for this charge in the quantum theory, and show that particles (created by $a^{s\dagger}_{\mathbf{p}}$) and antiparticles (created by $b^{s\dagger}_{\mathbf{p}}$) have opposite charges.

3. **Dirac Field and Causality:**

   (a) Using the explicit form of the Dirac equation solutions $u(\mathbf{p})$ and $v(\mathbf{p})$, prove the *spin sum identities*:

$$\begin{aligned} \sum_s u^s(\mathbf{p})\bar{u}^s(\mathbf{p}) &= \gamma \cdot p + m, \\ \sum_s v^s(\mathbf{p})\bar{v}^s(\mathbf{p}) &= \gamma \cdot p - m. \end{aligned} \tag{2}$$

(b) Consider a 4×4 matrix of Dirac field anticommutators,

$$F_{ab}(x,y) \;=\; \left\{\Psi_a(x), \bar{\Psi}_b(y)\right\}. \tag{3}$$

Expressing F_{ab} in terms of raising and lowering operators and using the spin sum identities, show that

$$F_{ab}(x,y) \;=\; (i\partial\!\!\!/ + m)_{ab}\, D(x-y), \tag{4}$$

where the function

$$D(x-y) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,\left(e^{-ip\cdot(x-y)} \;-\; e^{ip\cdot(x-y)}\right) \tag{5}$$

is familiar from our study of Klein-Gordon field.

(c) Use the expression (4) to argue that Ψ and $\bar{\Psi}$ *anticommute* at spacelike separations, that is, $F_{ab}(x,y) = 0$ for $(x-y)^2 < 0$.

(d) Consider two operators, $\mathcal{O}_1(x) = \bar{\Psi}_a(x)A_{ab}\Psi_b(x)$ and $\mathcal{O}_2(x) = \bar{\Psi}_a(x)B_{ab}\Psi_b(x)$, where A and B are some matrices. (It turns out that all operators corresponding to physically observable quantities of a fermion field have this generic form – consider for example the energy-momentum tensor and the Noether current encountered earlier in this homework.) Prove that $\mathcal{O}_1$ and $\mathcal{O}_2$ *commute* at spacelike separations:

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] \;=\; 0 \quad \text{at} \quad (x-y)^2 < 0, \tag{6}$$

as required by causality.