SUSY Breaking in Gauge Theories

Joshua Berger

With the Witten index constraint on SUSY breaking having been introduced in last week's Journal club, we proceed to explicitly determine the constraints on the Witten index in several important classes of gauge theories. We begin by going over the calculations necessary to understand Abelian gauge theory case in more depth, as this will be our template for the other calculations that we will do. In particular, we will generalize the method applied to Abelian gauge theories to discuss a large class of non-Abelian theories, briefly touching on some subtleties that can arise with non-standard gauge groups. Finally, we will discuss possible inclusions of matter (chiral) superfields and their effects on the analysis to the extent that they are understood. The discussion will be based on sections 6, 8, and 9 in Witten's paper [1]. References: Nucl. Phys. **B202**, 253 (1982); Weinberg, S. "The quantum theory of fields. Vol. 3: Supersymmetry."

I. INTRODUCTION AND BRIEF REVIEW

We begin these notes by giving a very brief overview of what we learned during the last Journal Club. We will then give a more detailed review of the proof that abelian gauge theories with vector-like matter (SQED) does not exhibit spontaneous SUSY breaking. After that, we will be ready to tackle the pure non-Abelian (SQCD) case and show that it too cannot break SUSY. Finally, we study SQCD with matter fields, where the answers are not so clear. These notes follow Witten's paper [1] quite closely. Some of the ideas are based on Weinberg Vol. 3 [2].

During the previous journal club, we studied some general constraints on spontaneous supersymmetry breaking in field theory. The primary constraint is related to a set of topological invariants based on the Witten index

$$\mathrm{Tr}(-1)^F. \tag{1}$$

We saw that a model can spontaneously break SUSY only if $\mathrm{Tr}(-1)^F = 0$. If $\mathrm{Tr}(-1)^F = 0$, however, it is not guaranteed that SUSY is broken. This index then allows us to determine that certain models cannot break SUSY.

The Witten index has certain nice properties. First and foremost, it is a topological invariant of the class of theories in that it is invariant under a large class of deformations of the parameters of the theory. In particular, we use the fact that if it is non-zero at small coupling in a finite volume theory, then SUSY will not be broken in the infinite volume or large coupling limits. Furthermore, the Witten index, properly regulated, is related to the difference in the number of bosonic and fermionic zero modes:

$$\mathrm{Tr}(-1)^F = n_B^{E=0} - n_F^{E=0}, \tag{2}$$

allowing it to be relatively easily calculated deep in the IR of a weakly coupled theory. It is insensitive to the UV completion of the theory. We can generalize the Witten index to any operator of the form

$$\mathrm{Tr}(-1)^F f(X), \tag{3}$$

where $f(X)$ is an arbitrary function of an operator X that commutes with the SUSY generators (and so is conserved). In essence, if the Witten index does not vanish in some corner of parameter space for some choice of $f(X)$, then, with some caveats, SUSY is not broken *anywhere* in parameter space. In particular, we can use $f(X)$ to project onto a portion of the Hilbert space with specific charge under X. This concludes our brief review of the salient features of the Witten index.

II. DETAILED REVIEW

A. Finite Volume Field Theory

We now review the more practical matters of actually calculating the Witten index and determining whether SUSY can be broken in a given theory. For these purposes, it is best to work at finite volume with periodic boundary conditions. If SUSY is unbroken, then we know that there is at least one zero energy state. If this is true at any finite volume, then it must be true in the infinite volume limit as the infinite volume limit of zero is zero.

Throughout most of these notes, we will work with plane-wave states constructed by using the free-field raising operators for the fields expanded around some choice of zero energy field configuration. As an example, when we looked at the Wess-Zumino model last time, we expanded the theory around the two choices of minima of the potential $\langle\phi\rangle = \pm m/2g$. We then found that there are plane-wave excitations about each of these vacua, all of which exhibit a mass gap. Such plane wave states, however, pose problems in an infinite volume field theory

Consider plane-waves in an infinite volume theory.

$$|p_1, s_1; p_2, s_2, \ldots p_n, s_n\rangle \sim a_{p_1}^{s_1\dagger} a_{p_2}^{s_2\dagger} \ldots a_{p_n}^{s_n\dagger}|0\rangle \tag{4}$$

These are not physical states, as they have infinitely well defined momentum. Technically, they are not normalizable:

$$\langle p_1, s_1; p_2, s_2, \ldots p_n, s_n|p_1, s_1; p_2, s_2, \ldots p_n, s_n\rangle \propto [\delta^{(4)}(0)]^n = \infty, \tag{5}$$

so that we cannot normalize the states to have unit length. The physical states are really wave-packets formed out of these plane waves and deep in the IR, we in principle have to worry about this fact. In particular, even for massless particles, the single particle plane-wave mode of a field with zero momentum and zero energy (henceforth called a zero mode excitation), is *not* a physical state in our Hilbert space. It must appear as some wave packet

$$|0\rangle_f = \int \frac{d^4p}{(2\pi)^4} f(p)|p\rangle, \qquad \int \frac{d^4p}{(2\pi)^4}|f(p)|^2 = 1, \qquad \int \frac{d^4p}{(2\pi)^4} p|f(p)|^2 = 0, \tag{6}$$

centered at $p = 0$, which may or may not have be an energy/momentum eigenstate with zero energy/momentum.

In the finite volume theory we can skirt around these technical issues as the allowed energy-momentum eigenstates are normalizable, physical states:

$$\langle p_1, s_1; p_2, s_2, \ldots p_n, s_n | p_1, s_1; p_2, s_2, \ldots p_n, s_n \rangle \propto \prod_i \delta_{p_i p_i} < \infty. \tag{7}$$

In particular, for massless excitations, there is no reason to exclude the state

$$a_0^{s_1 \dagger} |0\rangle \tag{8}$$

from the Hilbert space. Note that we can only guarantee the masslessness of the excitation if its mass is protected by some symmetry. For fermions, we are now guaranteed by the exclusion principle that there can be at most four zero energy states for each minimum about which we expand: no excitation, spin up excitation, spin down excitation, or excitation of both spins. For bosons, we have to work a bit more, but we hope that in finite volume we at least stand a chance of finding a finite number of discrete zero energy states. The hope of counting the zero energy states makes it essential that we do our calculations at finite volume.

B. Outline of General Program

In this section, we outline the general program that we will follow in studying all of the models below. Given a model, the difficult task is to determine the zero-energy, zero-momentum states in the Hilbert space at finite volume and weak coupling. The topological results that we derive using this information should then be extendable to strong coupling and infinite volume.

The first step in achieving this goal is to find the vacuum field configurations about which we can do our perturbation expansion. Since we are considering states with zero momentum, we will generally do this by considering field configurations that are constant in space. If the expansion about each of these vacuum configurations contains only massive excitations, as in the Wess-Zumino model, then we are done. If, on the other hand, the expansion about one or more of these vacuum configuration contains massless excitations, then we have more work to do. For fermions, we have to verify that all four zero mode excitations indeed have zero energy. For bosons, we have to hopefully find some way to resolve the degeneracy of the vacuum states. This means studying the interactions of the theory to determine if the excited states are lifted.

Once we have counted the number of fermionic and bosonic zero energy states, we need to devise a function $f(X)$, if such a function exists, to weigh our trace over the Witten operator by such that it does not vanish. We can then show conclusively that there is no SUSY breaking in the model at any point in parameter space.

C. Abelian SQED

We considered an Abelian gauge theory with vector-like (and therefore massive) chiral fields. We now go over the proof that such a model does not exhibit spontaneous supersymmetry breaking in detail as a benchmark example of applying the program outlined above. Note that all the statements we will make in this calculation implicitly assume that we are working in perturbation theory. For example, if we say that excitations have a mass gap, we assume that since perturbation theory is a good approximation, interactions do not alter this statement.

The gauge boson A and gaugino ψ are massless due to gauge symmetry. There are two equivalent ways of stating the consequences of this fact for our purposes, both of which will be useful. The first is that there are an infinite number of field vacua about which we can expand:

$$A^\mu = C^\mu, \qquad \psi = \epsilon, \tag{9}$$

where C^μ is a constant four vector and ϵ is a constant spinor. The other way of stating the consequence, as we have already done, is to say that, again neglecting interactions, we can apply an arbitrary number of creation operators to the vacuum state corresponding to any of these field vacua and still obtain a state with zero energy.

Let us first deal with the gauginos. As discussed earlier, there are at most four states involving excitations of the fermions about any given field vacuum. In this case, these four states all have zero energy in perturbation theory. To see this, note that perturbation theory implies that the vacuum involve no excitations of the heavy, vector-like matter fields. If this is the case, then the gaugino is exactly free and zero mode excitations really have zero energy. Thus, for each gauge field vacuum, there are four gaugino excitation vacua, two fermionic and two bosonic.

Next, we deal with the photons. In the infinite volume case, a constant gauge field is pure gauge: we can always remove a constant gauge field by a gauge transformation with $\alpha(x) = C\dot{x}$. We would like to pull the same trick at finite volume, but there is no guarantee that the gauge transformation will respect the boundary conditions. For the gauge field, this is not a problem since it will remain periodic regardless of the constant shift gauge transformation. Charged fermions, on the other hand, get multiplied by a space dependent phase $e^{iQec\cdot x}$ which need not be periodic. Whenever charge is quantized, however, we can guarantee that we can always shift the gauge field by an arbitrary integer multiple of some basic period given by requiring that an elementary charged fermion field be periodic in the gauge transformation:

$$\psi_e \to \exp(iec \cdot x)\psi_e = \psi_e. \tag{10}$$

This requirement means that we can shift by constants

$$C_i = \frac{2\pi n_i}{eL}, \tag{11}$$

where L is the period. Note that we are now working in a gauge $A_0 = 0$.

We complete the analysis by writing down a Hamiltonian describing zero-momentum field configurations. These are spatially constant configurations

$$h_i = \frac{1}{V}\int d^3x A_i. \tag{12}$$

We allow for a time dependence, since we would like to calculate the spectrum of zero-momentum modes. Note that the h_i must be periodic with period given by C_i in equation (11). The Lagrangian in terms of these modes is then

$$L = \frac{1}{2}\dot{h}_i^2 + \ldots, \tag{13}$$

where $\ldots$ represent terms with non-vanishing momentum and interactions. The Hamiltonian is then that of a free non-relativistic particle confined to a ring. We know the spectrum of such a system: there is a unique ground state with $E = 0$ given by wavefunction $\psi = 1$. The degeneracy is lifted and there is only one vacuum corresponding to excitations of the photon field.

In total, we have four vacuum states, 2 bosonic and 2 fermionic. Therefore, the basic Witten index $\mathrm{Tr}(-1)^F$ vanishes and provides no constraint on SUSY breaking. We need to pick an appropriate weighting operator. The simplest choice is to weigh by the charge conjugation operator C. Since gauge bosons are odd under charge conjugation, the gauginos must be as well. We therefore find that the fermionic vacua are odd and contribute a $+1$ to $\mathrm{Tr}(-1)^F C$ and the bosonic vacua are even and also contribute a $+1$. In total, we find that $\mathrm{Tr}(-1)^F C = 4 \neq 0$ and so SUSY cannot be broken in an Abelian gauge theory with vector-like matter for any coupling strength.

We will apply very similar techniques to analyze more general non-Abelian gauge theories.

III. NON-ABELIAN GAUGE THEORIES

Consider now a pure supersymmetric non-abelian gauge theory with a simple, non-Abelian Lie group G of rank r. In this section, we show that, in this model, $\mathrm{Tr}(-1)^F = r + 1$.

First, recall that the rank of a Lie group is the dimension of the Cartan subalgebra, which (in cases we care about) is the maximal set of generators that commute with each other. In the case of $SU(N)$, for example, the rank is $N-1$, as we can simultaneously diagonalize $N-1$ generators (σ^3 for $SU(2)$, λ^3 and λ^8 for $SU(3)$).

In such a theory, there are gauge fields and their superpartner gauginos that transform in the adjoint of the group:

$$A_\mu \to g(x)(A_\mu + i\partial_\mu)g^{-1}(x), \qquad \lambda \to g(x)\lambda g^{-1}(x), \tag{14}$$

for $g(x) \in G$. Since the gaugino transforms under the gauge group, it is coupled to the gauge field.

A. $SU(2)$ Gauge Theory

As a warm-up, let's consider the simplest non-Abelian gauge theory, $SU(2)$. We will work in a notation of $A_i = A_i^a t^a$, so that A_i are traceless, Hermitian matrices. We will work in a regime where the inverse length scale of the volume of our space is much larger than the strong coupling scale of the gauge theory so that the theory is weakly coupled at all scale available to us in our theory. The conclusions drawn based on the Witten index are of course independent of this kind of detail, so our result holds in the infinite volume case, which must have a strongly coupled regime.

We are looking for spatially constant solutions, since non-constant solutions will have non-zero momentum. We further want the energy to vanish, but the energy for the gauge fields alone is given by

$$H = \int d^3x \frac{1}{2}(\partial_0 A^i)^2 + \frac{1}{2}(F^{ij})^2. \tag{15}$$

We then need $F^{ij} = 0$, but $F^{ij} = \partial^i A^j - \partial^j A^i + g[A^i, A^j]$. The first two terms vanish since we have chosen A^i to be constant. We must further require that the commutator vanish. For $SU(2)$, up to a global gauge transformation, there is only one solution to this commutator vanishing:

$$A^i = C^i \sigma_3, \tag{16}$$

where C^i is an arbitrary vector of constants and σ_3 is the third Pauli matrix. At finite volume, at least, this solution is not pure gauge. Witten provides a cute argument for this, but we will soon see it explicitly anyway.

In this background field, there is a corresponding gaugino solution given by

$$\psi = \epsilon \sigma_3, \tag{17}$$

where ϵ is an arbitrary constant spinor. The Lagrangian contains a term given by $g[A^i, \psi]$, which arises in the covariant derivative of ψ. In order for this term to not contribute to the energy, ψ need to be parallel to A_i in $SU(2)$ space. Of course, the spinor needs to be spatially constant to have zero momentum.

For each vacuum of the C^i, there are vacua coming from the possible fermionic excitations. As long as the excitations are aligned with the direction in $SU(2)$ space of the background gauge field, they will have zero energy. There are four of them, as in the Abelian case.

We must next figure out what vacua exist for the gauge field zero mode excitations. We find that our gauge field configurations must be periodic just as in the abelian case. Although in that case, we relied on presumed charge quantization, in this case the periodicity is actually required for a consistent theory. The generators in any representation have quantized eigenvalues, so the "charge" of any state is quantized. Thus, the constraint no longer even depends on charge quantization. The fact that gauge field is periodic allows us to quantize its zero mode excitations on a compact space just and we find a unique vacuum.

So just as in the Abelian case, there are four vacua, two fermionic and two bosonic. But two of the states are not gauge invariant and must not be physical. To see this, consider the gauge transformation

$$g = i\sigma_2. \tag{18}$$

This transformation changes both the gauge field and the gaugino by a minus sign. For example,

$$C^i \sigma_3 \to C^i(-i\sigma_2)\sigma_3(i\sigma_2) = -C^i \sigma_3. \tag{19}$$

Thus, the fermionic states that we found have opposite sign under this transformation from the bosonic states. If we assume that the vacuum is even under g, then only the bosonic states are physical and $\mathrm{Tr}(-1)^F = 2$. If we assume that the vacuum is odd under g, the only the fermionic states are physical and $\mathrm{Tr}(-1)^F = -2$. Either way, the Witten index does not vanish and supersymmetry is not broken in a pure $SU(2)$ gauge theory.

B. Generic Non-Abelian Gauge Theory

We now generalize the analysis to an arbitrary simple gauge group (with some exceptions that we will note, but not discuss in detail). In order to find solutions to the equations of motion that have zero momentum and zero energy, we look for solutions that are spatially constant and have vanishing commutator. There are usually r such solutions: constant vectors proportional to the generators of the Cartan subalgebra. These objects commute with each other by definition. However, for the orthogonal group $O(N)$ with $N \geq 7$ and the exceptional groups, there are more solutions to this flatness condition. We will not discuss this situation further as it gets rather technical and the final physical result is the same: SUSY is not broken. A more complete discussion can be found in Ref. [3, 4]. So our gauge field solutions are

$$A_i = \sum_a C_i^a T^a, \tag{20}$$

where T^a lie in the Cartan subalgebra of the group and C_i^a are constants. Similarly, the gauginos must point in a direction in the Cartan subalgebra:

$$\psi_\alpha = \sum_a \epsilon_\alpha^a T^a, \tag{21}$$

where ϵ_α^a are constant spinors.

When we dealt with $SU(2)$, we noted that the physical states had to be invariant under a transformation g which mapped our set of vacua onto themselves. That is, it mapped the Cartan subalgebra of $SU(2)$, which only contains one element, onto itself. An equivalent way of saying this is that it was the transformation in $G = SU(2)$ that mapped the 1 dimensional torus generated by the Cartan subalgebra into itself. More generally, there is always a discrete subgroup of G that maps the torus generated by the Cartan subalgebra into itself, but does not commute with the generators of the Cartan subalgebra. Consider as an example the case of $SU(N)$. Since we can always find a basis where the generators of the Cartan subalgebra are diagonal, the Weyl group must be isomorphic to the group P_N of permutations of the N eigenvalues of the elements of the Cartan subalgebra for $SU(N)$. As before, the physical states must be invariant under the Weyl group.

By the same argument we made before, the C_i^a can be quantized as free periodic particles and have a unique ground state. Next, we have to deal with the fermionic degrees of freedom. As before, we can construct all possible states using our $2r$ creation operators. There are 2^{2r} such since for each creation operator we can construct a state that does or does not contain it. However, we have to project on the physical space of states that is invariant under the action of the Weyl group.

Consider first the state with no excitations $|\Omega\rangle$. This state is annihilated by all of the annihilation operators. It must transform in a one dimensional representation of the Weyl group. There are two such representations: the trivial one and the one where odd permutations give a $-$ sign. States which transform under trivial representation are called true invariants while states that transform under the nontrivial representation are called pseudo-invariants. The physical states must be true invariants, so Ω may not be a physical state. We do not know *a priori* whether it is or is not a true invariant, but it won't matter so we'll consider both cases. If it is a true invariant, all other physical states must be constructed by acting on it with true invariant operators. Otherwise, all the physical states must be constructed by acting on it with pseudo-invariant operators.

What invariant operators can we form? The building blocks are creation operators $a_\alpha^{a\dagger}$, which transform in r dimensional representations of the Weyl group. The relevant invariants can be found by noting that the Weyl group is a subgroup of the orthogonal group $O(r)$. For example, for $SU(N)$, the Weyl group is the permutation group of N elements, which can be written as the subgroup of $O(N-1)$ by considering the transformations that leave axes along the same lines. The true invariants of $O(N-1)$ is the Kronecker delta and the pseudo-invariant is the Levi-Civita symbol. Thus, the true invariant operators are of the form $U_{\alpha\beta} = a_\alpha^{a\dagger} a_\beta^{a\dagger}$ and the pseudo-invariant operators are of the form $V_{\alpha\beta\ldots} = \epsilon^{ab\cdots} a_\alpha^{a\dagger} a_\beta^{b\dagger}\ldots$. There are other invariants, but they vanish due to Fermi statistics. Using Fermi statistics, we can construct a unique U by contracting the spin indices anti-symmetrically. These states are all bosonic since an anti-symmetric contraction of the spins is spin 0. For the same reason, the V operators must be symmetric in their indices. They have spin $r/2$ (1/2 for each creation operator involved) and there are $r+1$ such operators (one for each possible z-axis spin eigenvalue, for example).

Next, we note that $U^{r+1} = 0$ by Fermi-Statistics and that V^2 can be expressed in terms of U^r by using Levi-Civita identities. For $|\Omega\rangle$ a true invariant, the physical vacua are composed of $U^n|\Omega\rangle$. There are $r+1$ such states and they are all bosonic, so that $\mathrm{Tr}(-1)^F = r+1$. For $|\Omega\rangle$ a pseudo-invariant, the physical vacua are composed of the states $V|\Omega\rangle$. There are $r+1$ of these states and they are all either fermionic or bosonic so that $\mathrm{Tr}(-1)^F = (-1)^{r+1}(r+1)$. In either case, it is always non-zero so SUSY is never broken.

C. Interpretation of the Vacua

Having show that SUSY is not broken in the case of a (nearly) generic pure non-Abelian gauge theory, we make one last note about the physical interpretation of the vacua we have found in terms of the chiral anomaly.

Consider an $SU(N)$ gauge theory. Classically, the Lagrangian for our non-abelian gauge theory is invariant under the chiral transformation $\psi \to e^{i\alpha}\psi$. This is the $U(1)_R$ of the SUSY theory. This symmetry, however, is well known to be anomalous. The gaugino transforms in the adjoint of $SU(N)$, which has an anomaly index $2N$. A chiral transformation then shifts the Yang-Mills angle by $-2N\alpha$. This transformation changes the action by a factor of

$$\Delta S = -2N\alpha \frac{1}{32\pi^2}\int d^4x F\tilde{F} = -2N\alpha n, \tag{22}$$

where we use an n instanton field in the last equality. Since our partition function goes like e^{iS}, the chiral transformation has no effect on the partition function if $-2N\alpha n$ is an integer multiple of 2π. This is the case for $\alpha = k\pi/N$. Thus, the anomaly breaks $U(1)_R \to Z_{2N}$ corresponding to all the chiral rotations $e^{ik\pi/N}$.

This symmetry is further broken since super Yang-Mills exhibits gaugino condensation and $\langle\psi\psi\rangle \neq 0$. The combination $\lambda\lambda$ is only invariant under the Z_2 subgroup of Z_{2N}, so the Z_{2N} is spontaneously broken to Z_2 and there are N vacua related by Z_{2N}/Z_2 transformations. These N vacua are essentially the N vacua we found at finite volume in our previous analysis.

To make this connection more concrete, we need to write linear combinations of the vacua that we found such that chiral transformations in Z_{2N}/Z_2 generate all possible independent vacua of the theory. If we suppose that $|\Omega\rangle$ is a true invariant and that $a^\dagger$ transforms with $U(1)_R$ charge $+1$, then U has chiral charge $+2$ and the vacua $U^n|\Omega\rangle$ have chiral charge $2n$. We then study the combination

$$|0\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} U^p |\Omega\rangle. \tag{23}$$

Applying a discrete chiral transformation $\alpha\pi n/N$, we get the n vacua

$$|n\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \exp(2\pi i n p/N) U^p |\Omega\rangle. \tag{24}$$

These are the N vacua obtained by spontaneous discrete chiral symmetry breaking.

This reasoning can be extended rather easily to $Sp(2N)$ gauge groups. For the problematic $O(N)$'s and the exceptional groups, however, more input is needed for we have not counted all the vacua.

IV. GAUGE THEORIES WITH MATTER

In many cases there is little we can say about gauge theories with chiral matter fields. We can divide such theories into several classes:

1. Gauge theories with massive vector-like matter;
2. Gauge theories with massless matter in real representations;
3. Gauge theories with massless chiral matter;
4. Gauge theories with moduli (i.e. scalars with flat bare potentials).

Case 1 is trivial since excitations of these fields will always have a gap. Case 2 is rather easy to deal with since we could in principle write down a mass term for such fields and not violate any gauge symmetries. We can therefore calculate the index in the massive case, where we know it does not vanish, and take the limit $m \to 0$ provided this limit is well defined. Case 3 is the most non-trivial as there is no massive limit to take and we will not even discuss it.

We now turn our attention to case 4. There are several possibilities for the effect of quantum corrections in this case. The potential could remain flat, get lifted to a minimum that does ($V \neq 0$) or does not ($V = 0$) break supersymmetry, or get a potential that tends toward a constant zero or non-zero value at large field. We would like to determine whether the Witten index can help us resolve this question.

If the potential remains flat, then the Witten index is not well defined as there is an exact continuum of zero energy sates. If not, then we can calculate the Witten index, but we have learned nothing about the non-perturbative physics from it. If we can add a term to the potential that lifts the degeneracy, then we can calculate the Witten index with the coupling turned on and take the limit as it goes to zero. This is rarely possible.

We conclude that there is very little we can do in the case of massless matter fields when we cannot add a term that lifts the potential. This is a very strong limitation on the usefulness of the Witten index.

[1] E. Witten, Nucl. Phys. **B202**, 253 (1982).
[2] S. Weinberg, Cambridge, UK: Univ. Pr. (2000) 419 p.
[3] E. Witten, JHEP **9802**, 006 (1998). [hep-th/9712028].
[4] V. G. Kac and A. V. Smilga, arXiv:hep-th/9902029.