~~Also~~

The duality is a weak-strong duality in the sense that when $\lambda \gg 1$, the gravity side is weakly curved and so α' corrections are small. On the other hand, when $\lambda \ll 1$, the gauge theory is weakly coupled but α' effects are important.

Hence for the duality to always hold, ST must come in. We will later show that even for $\lambda \gg 1$, the duality is not complete without ST.

Here's what we've learned so far

$$\text{IIB ST on } AdS^5 \times S^5 \iff \mathcal{N}_4 = 4 \ SU(N) \text{ SYM}$$

$$\frac{\alpha'}{L^2} \iff \lambda$$

$$g_s e^{\phi} \iff g_{YM}$$

III.A - Matching symmetries

An important check of the duality is the matching of symmetries. A conformal ~~group~~ field theory is invariant under

1. Lorentz transformations $x^\mu \mapsto x^\mu + \omega^\mu{}_\nu x^\nu$
2. Translations $x^\mu \mapsto x^\mu + a^\mu$
3. Dilations $x^\mu \mapsto x^\mu + \lambda x^\mu$
4. Special transformations $x^\mu \mapsto x^\mu + x^2 b^\mu - 2 b\cdot x\, x^\mu$

The resulting group is $SO(4,2)$. (*Note*: In $d=2$ the group is much larger).

This matches onto the $SO(4,2)$ isometry of AdS^5. The Poincaré subgroup corresponds to that of the CFT. We have also

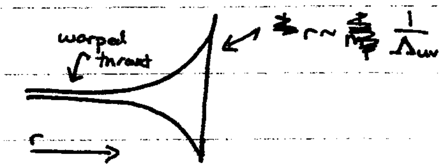
$$(x^\mu, r) \mapsto (\lambda x^\mu, \lambda^{-1} r) \qquad [20]$$

This corresponds to dilation in the CFT. Since RG flow is in essence a description of how systems behave under coordinate rescaling, we have

$$\text{translation in } r \leftrightarrow \text{RG flow} \qquad [21]$$

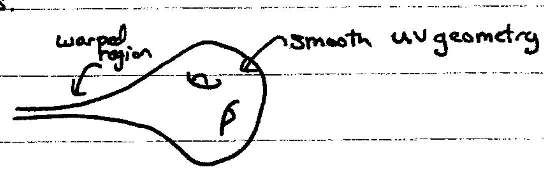
r is often called the holographic direction

Let's pursue this a little further. Suppose that we wish to couple our system to gravity. There ~~are~~ is significant evidence to suggest that there is a sense of minimal length. Hence the ~~a~~ gravity dual of a theory with gravity must end at some r_{max} (similar to RS II models).



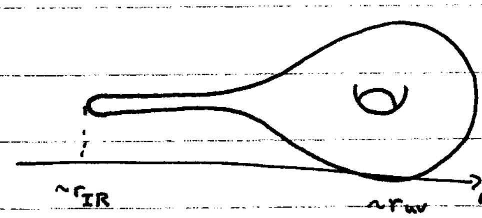
It is essentially a fact that in ST there is no way to end the geometry in a way that preserves isometries.

Hence the metric must deviate at large r, from AdS$_5\times S^5$. Similarly, the theory with gravity differs ~~precisely~~ from the theory without gravity by irrelevant operators. Hence ~~roughly~~ (roughly)

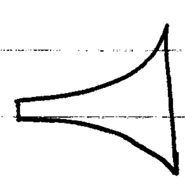


$$\text{UV localized modes} \sim \text{irrelevant operators.}$$

Similarly if the theory exhibits confinement at some low dynamical scale we must have a smooth IR ~~cap~~ end.



This is reminiscent of RSI



$\mathcal{N}_4=4$ field theories exhibit an $SU(4) \sim SO(6)$ R-symmetry. Similarly, S^5 has an $SO(6)$ isometry. Now, because of the above deviation from $AdS^5 \times S^5$, $SO(6)$ is not an isometry of the bulk geometry.

This is consistent with the folk-theorem that quantum gravity does not admit global symmetries. There are two ways that an apparent global symmetry can result in the IR

1. It is an accidental symmetry – "emergent symmetry"
2. It is secretly "weakly gauged" – "flavor symmetry"

Thus,

$$\text{isometry} \iff \text{emergent global symmetry}$$ [22] [?]

$\mathcal{N}_4=4$ exhibits Montonen-Duality

$$\tau \sim \theta_{YM} + \frac{i}{g_{YM}^2} \to \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$ [23]

This is exhibited also by IIB ST

$$\tau = C_0 + i e^{-\phi} \to \frac{a\tau+b}{c\tau+d}$$ [24]

III.B – Correlation functions and the matching of states

Let's consider a massive scalar field in AdS^5. This could follow from, e.g. dimensional reduction on S^5. Defining $u = \frac{L^2}{r^2}$ the AdS^5 metric is

$$ds_5^2 = \frac{1}{u}\, dx_4^2 + \frac{L^2}{4u^2}\, du^2 \qquad [25]$$

The action for a massive scalar field is

$$S = -\frac{1}{2}\int d^5x \sqrt{\tilde{g}} \left\{ \tilde{g}^{mp}\, \partial_m \Sigma\, \partial_p \Sigma + m^2 \Sigma^2 \right\} \qquad [26]$$

Note that ~~a~~ tachyons in AdS^{d+1} are stable if (Breitenlohner, Freedman)

$$m^2 \geq \frac{-d^2}{4} \qquad [27]$$

From here on, we work in Euclideanized space. ~~Write~~ In ~~Fourier space~~ 4d Fourier space the equation of motion is

$$0 = 4u^2 \partial_u^2 \Sigma - 4u\, \partial_u \Sigma - \mu^2 \Sigma - u K^2 \Sigma \qquad \mu^2 = L^2 m^2 \quad K^2 = L^2 k^2 \qquad [28]$$

At ~~large~~ small u, this is solved by

$$\Sigma = \sigma_1 u^{1-\frac{\nu}{2}} + \sigma_2 u^{1+\frac{\nu}{2}} \qquad [29]$$

$$\nu = \sqrt{4+\mu^2}$$

For a scalar saturating [26], the solutions are u and $u \log u$.

The on-shell action is then a boundary term

$$S = -\frac{1}{L}\int d^4k\, \frac{1}{u}\, \partial_u \Sigma\, \Sigma \Big|_{u=0}^{u=\infty} \frac{d^4k}{(2\pi)^4} \qquad [30]$$

Now, the most natural boundary condition to impose at $u=\infty$ is $\partial_u \Sigma = 0$. This ensures that the solution is smooth at the origin. Then

$$S = \frac{1}{L}\int d^4x \left\{ \left(1-\frac{\nu}{2}\right) \sigma_1^2 u^{-\nu} + 2\sigma_1 \sigma_2 + \sigma_2^2 \left(1+\frac{\nu}{2}\right) u^{\nu} \right\}_{u=0} \frac{d^4k}{(2\pi)^4} \qquad [31]$$

This is divergent even after the infinite volume of R^4 is factored out. We could get ~~an~~ a finite result by imposing Neumann conditions at the $u=0$ boundary, but this just gives zero which is not very interesting. Instead we will impose Dirichlet conditions

$$\Sigma \to \hat{\sigma}_1(k)\, u^{1-\frac{\nu}{2}} \qquad [32]$$

We can also handle the divergence through a process called <u>holographic renormalization</u> (Skenderis, ...) Define a regulated action

$$S_{reg} = -\frac{1}{2}\int_{u>\epsilon} d^5x \sqrt{\tilde{g}} \left\{ \tilde{g}^{mp} \partial_m \Sigma \partial_p \Sigma + m^2 \Sigma \right\} \qquad [33]$$

The divergence can be removed by the addition of a counterterm Lagrangian

$$S_{ct} = -\frac{1-\frac{\nu}{2}}{L}\int d^4x \sqrt{-\det(\tilde{g}_{\mu\nu})}\, \Sigma^2 \Big|_{u=\epsilon}$$

$$= -\frac{1-\frac{\nu}{2}}{L}\int \frac{d^4k}{(2\pi)^d}\left\{ \sigma_1^2 u^{-\nu} + 2\sigma_1\sigma_2 + \sigma_2^2 u^{\nu} \right\}\Big|_{u=\epsilon} \qquad [34]$$

The subtracted action is then

$$S_{sub} = S_{reg} + S_{ct} \qquad [35]$$

Finally the renormalized action is

$$S_{ren} = \lim_{\epsilon \to 0} S_{reg}$$

$$= \frac{\nu}{L}\int \frac{d^4k}{(2\pi)^4}\, \sigma_1(k)\,\sigma_2(k) \qquad [36]$$

Now, let's ~~consider~~ see what happens if we vary the boundary conditions. Then

$$\frac{\delta S}{\delta \sigma_1} \sim \sigma_2 \qquad [37]$$

Then,

$$\frac{\delta^2 S}{\delta \sigma_1^2} \sim \frac{\delta \sigma_2}{\delta \sigma_1} \qquad [38]$$

To proceed, let's ~~work~~ solve the equation of motion for all u. The solution is

$$\Sigma = c_1 u\, I_\nu(k\sqrt{u}) + c_2 K_\nu(k\sqrt{u}) \qquad [39]$$

where I_α and K_α are modified Bessel functions. Imposing regularity as $u \to \infty$, sets $c_1 = 0$. ~~Then~~

$$\Sigma'(u \to \infty) \to 0 \qquad [40]$$

As $u \to 0$,

$$\Sigma\sigma \to c_2\left\{\Gamma(-\nu)K^{\nu}2^{-1-\nu}u^{1+\frac{\nu}{2}} + \Gamma(\nu)K^{-\nu}2^{-1+\nu}u^{1-\frac{\nu}{2}}\right\} \qquad [41]$$

Hence

$$\sigma_1 = c_2\Gamma(\nu)\left(\frac{2}{K}\right)^{\nu}\frac{1}{2} \qquad \sigma_2 = c_2\Gamma(-\nu)\left(\frac{K}{2}\right)^{\nu}\frac{1}{2}$$

$$\Rightarrow \sigma_2 = \sigma_1\left(\frac{K}{2}\right)^{2\nu}\frac{\Gamma(-\nu)}{\Gamma(\nu)}$$

$$\Rightarrow \cancel{\frac{\delta\sigma_2}{\delta\sigma_1}} \quad \cancel{\frac{\delta\sigma_2(K)}{\delta\sigma_1(-K)}} \quad \frac{\delta\sigma_2}{\delta\sigma_1} \sim \left(\frac{K}{2}\right)^{2\nu} \qquad [42]$$

~~A more careful~~

There is a momentum conserving δ-function that has been suppressed.

Fourier transforming,

$$\frac{\delta\sigma(x)}{\delta\sigma(y)} \sim \frac{1}{(x-y)^{2\nu+4}} \qquad [43]$$

This is precisely the two-point function of a scalar operator with dimension

$$\Delta = 2+\nu = 2+\sqrt{4+\mu^2} \qquad [44]$$

in a CFT!

Hence we have

$$\text{field} \Leftrightarrow \text{operator} \qquad [45]$$

$$\text{mass} \Leftrightarrow \text{dimension}$$

Moreover,

$$\text{dominant mode} \Leftrightarrow \delta\mathcal{Z} \sim \mathcal{O}\sigma_1 \qquad [46]$$

$$\text{subdominant mode} \Leftrightarrow \langle\mathcal{O}\rangle \sim \sigma_2$$

And, at least in this limit

$$e^{I_{\text{sugra}}} = e^{W} \qquad [47]$$

classical sugra action — generator of connected Green's function

Dimensional reduction gives $m^2 \sim \frac{\#}{L^2}$. That is, Δ is independent of λ. This is non-generic and ~~thus~~ is a consequence of the fact that these fields ~~have a~~ correspond to operators with protected dimensions.

However, there are other modes coming from stringy excitations

$$m^2 \sim \frac{1}{\alpha'} \qquad [48]$$

Then
$$\Delta \sim \frac{L}{\alpha'^{1/2}}$$
$$\sim (g_s N)^{1/4}$$
$$\Delta \sim \lambda^{1/4} \qquad [49]$$

Without ST, there would be no way to get those modes of such large dimensions which ~~are generica~~ are to be expected. That is, the duality cannot be complete without ST.

The matching of modes has largely been done, but we won't go through it here.

Now suppose AdS^5 suddenly ends at ~~r~~ $u = u_{IR} < \infty$. As discussed previously this corresponds to ~~the~~ a confinement scale. We have

$$\Sigma = c_1 u\, I_\nu(k\sqrt{u}) + c_2 u\, K_\nu(k\sqrt{u}) \qquad [50]$$

The Neumann condition gives

$$c_1 = -c_2 \frac{(\nu-4) K_\nu(k\sqrt{u_{IR}}) + \sqrt{u_{IR}}\, k\, K_{\nu-1}(k\sqrt{u_{IR}})}{(\nu-4) I_\nu(k\sqrt{u_{IR}}) - \sqrt{u_{IR}}\, k\, I_{\nu-1}(k\sqrt{u_{IR}})} \qquad [51]$$

~~and~~

$\Sigma^5 \sim$

a non-trivial c_1. This transfers ultimately to poles in the two-point functions, as expected. The poles correspond to glueballs.

The gravitational theory also has interactions. They give rise to higher-n functions. The prescription is (roughly)

$$\langle \mathcal{O} \cdots \mathcal{O} \rangle \sim \left. \frac{\delta^n I_{sugra}}{\delta \sigma_1^{\,n}} \right|_{\sigma_1 = 0} \qquad [52]$$

IV. ~~Kd~~ A gravity dual of an $\mathcal{N}_4=1$ theory

The ~~above case~~ example of $AdS^5 \times S^5$ is important and the best understood, but for many reasons we may be interested in examples with less supersymmetry. Recall

$$AdS^5 \times S^5 \simeq R^{3,1} \times_w R^6 \qquad [53]$$

The large amount of symmetry is a result of the large symmetry of R^6.

It is a famous fact ~~that~~ (Candelas, Horowitz, Strominger, Witten) that generic Calabi-Yau 3-folds are $\frac{1}{4}$ BPS. A Calabi-Yau n-fold is a 2n-dimensional space such that

1. It is Kähler, $ds_6^2 = 2 g_{\mu\bar\nu} dz^\mu d\bar z^{\bar\nu}$ $\quad g_{\mu\bar\nu} = \partial_\mu \bar\partial_{\bar\nu} \mathcal{K}(z,\bar z)$ $\quad [54]$
2. It is Ricci-flat $\quad R_{mp}=0$

$R^6 \simeq C^3$ is such a space but is non-generic.

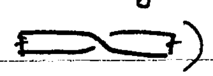
A special class of Calabi-Yaus are cones. A cone essentially means the metric is (for 6d)

$$ds_6^2 = dr^2 + r^2 ds_{X^5}^2 \qquad [55]$$

where X^5 is a compact metric. For the 6d space to be Ricci flat, X^5 must be Einstein ($\check R_{\theta\varphi} \propto \check g_{\theta\varphi}$). If the 6d space is in addition ~~C~~ Calabi-Yau then X^5 is called Sasaki.

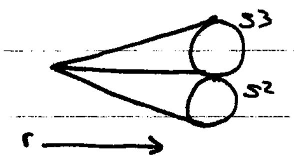
A famous Sasaki-Einstein space is $T^{1,1}$. $T^{1,1}$ is a $U(1)$ fibration over $S^2 \times S^2$

(Aside on fiber bundles: A fiber bundle is a generalization of a direct product. Schematically, X is fibered over Y if locally the total space is $X \times Y$, but X changes over Y. Eg, $S^3 \to S^2$

~~Mobius strip~~ $\begin{matrix} I \to \text{Möbius strip} \\ \downarrow \\ S^1 \end{matrix}$ )

It is also an S^2 fibered over S^3. $\quad S^2 \to T^{1,1} \downarrow S^3$.

The picture we ~~have to~~ should have in mind is



The metric on $T^{1,1}$ is

$$ds_5^2 = \frac{1}{9}\left(d\psi + \cos\theta^1 d\varphi^1 + \cos\theta^2 d\varphi^2\right)^2 + \frac{1}{6}\sum_{i=1}^{2}\left(d\theta^{i2} + \sin^2\theta^i d\varphi^{i2}\right) \qquad [56]$$

~~placing N D3 branes~~

The resulting 6d Calabi-Yau is called the conifold ~~[illegible]~~ Can (see: Candelas, de la Ossa). It is the locus in $\mathbb{C}^4$ satisfying

$$z^A z^A = 0 \qquad z^A \bar{z}^{\bar{A}} \propto r^3$$

The point at $z=0$ is a conifold singularity. It is resolved by stringy effects.

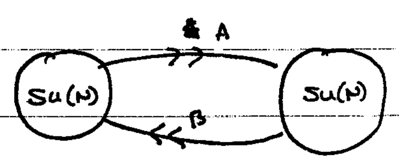
Now suppose that we place N D3 brans at the tip of the conifold. The backreaction takes the form (Klebanov, Witten)

$$ds_{10}^2 = e^{2A} dx_4^2 + e^{-2A}\left(dr^2 + r^2 ds^2_{T^{1,1}}\right)$$

$$e^{-4A} = 1 + \frac{L^4}{r^4} \qquad L^4 = \frac{4\pi^4 g_s N \alpha'^2}{V_{T^{1,1}}} \qquad V_{T^{1,1}} = \frac{16\pi^3}{27} \qquad [57]$$

We can once more take a low energy limit. The resulting geometry is $AdS^5 \times T^{1,1}$

Unlike $\mathbb{C}^3$, the conifold preserves only 4 (regular) supercharges. Analysis of the D3 theory shows that this should be dual to



[58]

$$W = \frac{\lambda}{2}\epsilon^{ij}\epsilon^{kl}\,\mathrm{tr}\left(A_i B_k A_j B_l\right)$$