$\boxed{\text{An introduction to AdS/CFT and Klebanov-Strassler}}$

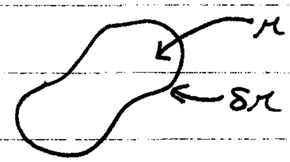
I. Warping and the holographic principle

Let's consider a ~~black~~ Schwarzchild black hole in 4d. It carries a mass M and has - in the classical limit - an event horizon of radius $R_S \propto M$. One can argue (Bekenstein) that the entropy of the BH is

$$S = \frac{A}{4G_N\hbar} \qquad [1]$$

~~The~~ Roughly, the entropy is a measure of the information content. One might have naïvely expected the entropy to grow with the volume and not the area.

This is the basis of the holographic principle ('t Hooft, Susskind) which roughly states that ~~the~~ a gravitational theory in a spacetime region $\mathcal{M}$ is encapsulated by ~~phys~~ a theory on the boundary $\delta\mathcal{M}$.



The most precise implentation of this is the gauge/gravity correspondence. The correspondence consists of pairs of the type

$$\text{gravity on } \mathbb{R}^{d-1,1} \times_w X^q \longleftrightarrow \text{gauge theory on } \mathbb{R}^{d-1,1} \qquad [2]$$

Clarification of "$\longleftrightarrow$" is one of the goals of these lectures.

$\times_w$ denotes a warped product. It means that the spacetime isn't a direct product but that the metric takes the form (mostly positive signature)

$$ds_D^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2\gamma A(y)} g_{mn}(y)dy^m dy^n \qquad [3]$$

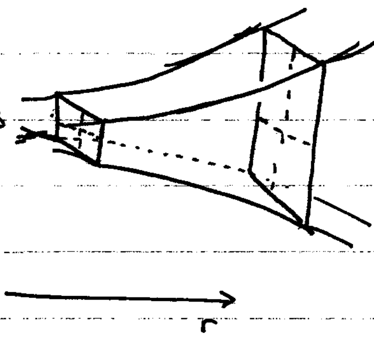
$g_{mn}(y)$ is a metric on X^q and $\eta_{\mu\nu}$ is the flat d-dimensional Minkowksi direction. γ is some convenient constant. $A(y)$ (or e^{nA}) is called the warp factor.

A particularly important example is anti-de Sitter space (AdS). ~~AdS^{d+1} is~~

$$AdS^{d+1} = R^{d-1} \times_w R^{\geq 0}$$

$$ds^2_{d+1} = \left(\frac{L}{r}\right)^2 dx_d^2 + \left(\frac{r}{L}\right)^2 dr^2 \qquad [4]$$

$y = e^{-r/L}$ and $z = \frac{L}{r}$ are often used. AdS has many interesting properties, few of which we will use here, but are important for a deeper understanding of the duality.



AdS plays a central role in the correspondence.

Example:

$$\text{~~AdS~~ type IIB supergravity on } AdS^5 \times S^5 = R^{3,1} \times_w R^6 \longleftrightarrow \mathcal{N}_4 = 4 \ SU(N) \ SYM \qquad [5]$$

$\mathcal{N}_4 = 4$ SU(N) SYM is an example of a conformal theory (i.e. one invariant under all angle-preserving transformations including coordinate rescaling). More generally,

$$\text{gravity on } AdS^{d+1} \times X^p \longleftrightarrow \text{CFT on } R^{d-1,1} \qquad [6]$$

These examples ~~are~~ go under the name AdS/CFT. The term is often used as a metonym for the ~~c~~ correspondence as a whole.

My personal bias is toward phenomenological applications, so I will focus on $d=4$ examples. There is rich physics to be found with other d.

If sufficently limited questions are asked, we do not need ~~string theory for~~ quantum gravity. However, I will consider stringy examples because

1. The duality is most clearly "derived" from strings.
2. The duality cannot possibly be exact with classical gravity
3. Personal bias.

II. Elements of string theory

For $d=4$, type-IIB string theory works best (though Sakai-Sugimoto, an attempt at AdS/QCD, uses IIA). String theory isn't a spacetime field theory, but at low energies and weak coupling it has an effective field theory description. For type-IIB string theory, the eft is type-IIB supergravity.

IIB sugra is the $\mathcal{N}_{10}=(0,2)$ theory. Its ~~___~~ field content is ~~the~~ completely fixed by supersymmetry:

metric	$\hat{g}_{MP}$	symmetric tensor	Neveu-Schwarz-Neveu Schwarz sector
Kalb-Ramond field	$\hat{B}_{MP}$	anti-symmetric tensor	
dilaton	ϕ	scalar	

~~dil~~ axion	C	scalar	Ramond-Ramond sector
RR 2-form potential	C_{MP}	anti-symmetric tensor	
RR 4-form potential	C_{MPNS}	anti-symmetric tensor	

gravitini	$\Psi_M^{i=1,2}$	left-handed Majorana-Weyl-Rarita-Schwinger	NS-R sector
dilatini	$\chi^{i=1,2}$	left-handed Majorana-Weyl	

These modes come from the lightest excitations of closed strings. ST has an infinite tower of modes with mass splittings set by the string tension

$$\tau_{F1} \sim \frac{1}{\alpha'} \qquad \Delta m^2 \sim \frac{1}{\alpha'} \sim \frac{1}{\ell_s^2} \tag{7}$$

The coupling strength of strings is set dynamically.

$$g_{IIB} = g_s e^{\phi} \tag{8}$$

(NB: notational differences differ greatly here)

To get from ST to sugra, we take the limits in the following way

$$\begin{array}{c} \text{Quantum string theory} \\ \downarrow \; g_{IIB} \to 0 \\ \text{Classical string theory} \\ \downarrow \; p^2\alpha' \to 0 \\ \text{Classical supergravity} \end{array}$$

Note: M-theory and F-theory gives us a way to take $p^2\alpha' \to 0$ before $g \to 0$.

II.A. Black p-branes

Consider Einstein-Maxwell theory in 4d. The equations of motion admit solutions for massive charged black holes (Reissner-Nordström). They admit a particularly limit called the extremal limit in which $M = Q$ in some appropriate units. These branes are then stable — Hawking radiation will no longer occur.

The type-II theories have analogues to the Maxwell fields, namely the RR and NS potentials.

A black p-brane is analogous to a charged black-hole; a black Dp-brane will couple electrically to $C_{(p+1)}$ for $p \leq 3$ and magnetically to $C_{(7-p)}$ for $p \geq 3$. In IIB, we hence have stable ~~branes~~ black p-branes for odd p (In IIA we have even p). ~~The electric~~ charge

The black 3-brane solution is (Townsend; Horowitz, Strominger)

$$ds_{10}^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A}(dr^2 + r^2 ds_{S^5}^2)$$

$$e^{\phi} = 1 \qquad [9]$$

$$C_{\mu\nu\rho\sigma} = g_s^{-1} e^{4A} \varepsilon_{\mu\nu\rho\sigma}$$

$$e^{-4A} = 1 + \frac{L^4}{r^4} \qquad L^4 = 4\pi g_s N \alpha'^2$$

Note that a D3 couples both electrically and magnetically to C_4, so there are additional components of C_4 that cannot be globally defined (c.f. - Dirac string for a magnetic monopole in 4d).

N is the amount of D3 charge. Recall in the Maxwell theory that the amount of charge in a region is

$$Q \propto \int_{\delta\Omega} \vec{E}\cdot d\vec{A} \sim \int_{\delta\Omega} F_{ij}\, dx^i dx^j \sim \int F_{(2)} \qquad [10]$$

Analogously, we define

$$F_{MNPRS} \cong \partial_{[M} C_{NPRS]} \qquad [11]$$

Then,

$$N = \frac{1}{4\pi^2 \alpha'^2}\int F_{(5)} \qquad [12]$$

Black p-branes can be thought of as solitonic configurations of closed strings.

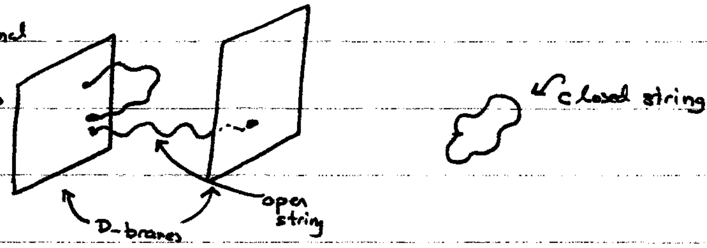
It is possible to consider finite temperature black branes. These are very relevant for applications to condensed-matter and heavy ion physics, but we will not use them here.

Black p-branes are "half-BPS." This means many things, but here I mean p-branes that it preserves half of the supersymmetries. That is, IIB has 32 supercharges, but only half of them annihilate the state corresponding to a black Dp-brane.

II.B. - Dp-branes

In addition to closed strings, type-II theories contain open strings.

These end on $(p+1)$-dimensional surfaces called Dp-branes (the D is for Dirichlet) (Dai, Leigh, Polchinski).



The lightest modes of ~~a string~~ an open strings on a coincident stack of N Dp-branes in flat space give a $(p+1)$-dimensional $U(N)$ SYM theory with 16 supercharges.

For $p=3$, this is $\mathcal{N}_4 = 4$ SYM.

vector multiplet — $\mathcal{N}_4=4$

vector multiplet, adjoint hypermultiplet — $\mathcal{N}_4 = 2$

vector multiplet, adjoint chiral multiplet, 2 adjoint chiral multiplets — $\mathcal{N}_4 = 1$

vector, adjoint Weyl, 3 adjoint scalars, 3 adjoint Weyl — $\mathcal{N}_4 = 0$

In $\mathcal{N}_4=1$ language, the superpotential is

$$W \sim g_{YM}\, \varepsilon_{ijk}\, \Phi^i \Phi^j \Phi^k \qquad [13]$$

There are also many massive open-string states.

The coupling constant for a D3 is $\sim g_s e^{\phi} \sim g_{YM}^2$

Let's recall that g_{YM} is not the correct expansion parameter for Yang-Mills.

Instead, loops involve summing over colors and so the correct expansion is the 't Hooft parameter

$$\lambda = g_{YM}^2 N \qquad [14]$$

The scalars on the D3-world volume correspond to transverse fluctuations of the Dp-brane. That is, a Dp-brane is a dynamical object and the dynamics are described by open strings theory.

A non-trivial fact is that Dp-branes carry Ramond-Ramond charge (Polchinski). ~~The most sensible interpretation is that~~ In fact the charge and tension match those of a black p-brane. The most sensible interpretation is that D-branes and black p-branes are the same thing.

String theory is not a theory of strings alone, but a theory of strings and branes. Indeed there are a number of dualities between string theories that map strings and branes to each other so that there is no invariant distinction between them.

III. Basics of AdS/CFT

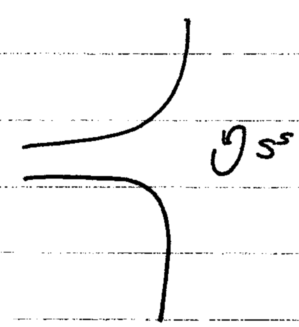
Let's return to the black 3-brane picture

$$ds_{10}^2 = \left(1+\frac{L^2}{r^2}\right)^{-1} dx_4^2 + \left(1+\frac{L^2}{r^2}\right)\left(dr^2 + r^2 ds_{S^5}^2\right) \qquad [15]$$

Note that in writing this picture we have implicitly fixed the position of the 3-branes. There are two interesting regions.

When $r \gg L$, the space is approximately flat $R^{9,1}$.

When $r \ll L$, it is strongly warped $AdS^5 \times S^5$.



$$e^{-4A} \to \begin{cases} 1 & r \gg L \\ \dfrac{L^4}{r^4} & r \ll L \end{cases}$$

~~[illegible]~~

One can show (Maldacena) that in the low-energy limit these two sectors decouple. Moreover since we have fixed the positions of the 3-branes, there are no intermediate modes. Hence in the black-brane picture we have that theory is

$$\text{IIB sugra on } AdS^5 \times S^5 \quad \otimes \quad \text{IIB sugra on } \mathbb{R}^{9,1} \qquad [16]$$

~~Now let's consider~~

Now let's consider the open-string picture. The gauge theory ~~f~~ is

$$U(N) \approx SU(N) \times U(1) \qquad [17]$$

The $U(1)$ corresponds to fluctuations of the center-of-mass and so by fixing the position of the D3s we essentially just ignore the $U(1)$. The theory consists of open strings on the D3s and closed strings in $\mathbb{R}^{9,1}$. In the low energy limit, there are no interactions between them and so the theory is

$$\mathcal{N}_4 = 4 \ SU(N) \text{ Yang Mills} \quad \otimes \quad \text{IIB sugra on } \mathbb{R}^{9,1} \qquad [18]$$

Since black ~~p~~ 3-branes are D3-branes we must have at low energies

$$\boxed{\text{IIB sugra on } AdS^5 \times S^5 \iff \mathcal{N}_4 = 4 \ SU(N) \text{ SYM}} \qquad [19]$$

Now let's make a couple of observations:

1. $\mathcal{N}_4 = 4$ $SU(N)$ SYM is a CFT so there is no meaning to ~~"[illegible]~~ the phrase "low energy"
2. $AdS^5 \times S^5$ is a valid solution to IIB eom without ever having to make the decoupling argument.

We thus ~~suppose~~ conclude that [19] must hold at all energies.

~~Here~~

The duality is a weak-strong duality in the sense that when $\lambda \gg 1$ the gravity side is weakly curved and so α' corrections are small. On the other hand, when $\lambda \ll 1$ the gauge theory is weakly coupled but α' effects are important.

Hence for the duality to always hold, ST must come in. We will later show that even for $\lambda \gg 1$, the duality is not complete without ST.

Here's what we've learned so far

$$\text{IIB ST on } AdS^5 \times S^5 \Leftrightarrow \mathcal{N}_4 = 4 \; SU(N) \; SYM$$

$$\frac{\alpha'}{L^2} \Leftrightarrow \lambda$$

$$g_s e^{\phi} \Leftrightarrow g_{YM}$$

III.A - Matching symmetries

An important check of the duality is the matching of symmetries. A conformal ~~group~~ field theory is invariant under

1. Lorentz transformations $x^\mu \mapsto x^\mu + \omega^\mu{}_\nu x^\nu$
2. Translations $x^\mu \mapsto x^\mu + a^\mu$
3. Dilations $x^\mu \mapsto x^\mu + \lambda x^\mu$
4. Special transformations $x^\mu \mapsto x^\mu + x^2 b^\mu - 2 b\cdot x\, x^\mu$

The resulting group is $SO(4,2)$. (Note: In $d=2$ the group is much larger).

This matches onto the $SO(4,2)$ isometry of AdS^5. The Poincaré subgroup corresponds to that of the CFT. We have also

$$(x^\mu, r) \mapsto (\lambda x^\mu, \lambda^{-1} r) \qquad [20]$$