Strongly coupled gauge theory

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Abstract

In this report, we discuss the physics of strongly coupled gauge theory, from ordinary space-time to super-space. Firstly we briefly review the topological objects of Yang-Mills theory in ordinary space-time, where we focus on the electric-magnetic duality. Later on we extend our discussion to Super Yang-Mills with matter added, where we discuss holomorphic gauge couplings and Seiberg duality. Finally we discuss Seiberg-Witten theory. The application to gauge-mediated dynamical SUSY breaking based on MSSM is straightforward. In above discussion, we will mainly focus on SUSY with $N \leq 2$. [3][4]

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I. TOPOLOGICAL FIELD THEORY IN PARTICLE PHYSICS

A. 1+1 Dimension solitons

We have the experience in 1+1 Dimensions for a real scalar field theories with a quartic or sine-Gordon potential.

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\partial_x \phi)^2 - V(\phi) \tag{1}$$

where

$$V(\phi) = \frac{\lambda}{4!}(\phi^2 - v^2)^2, \quad \textbf{or} \quad V(\phi) = A(1 - \cos\frac{2\pi\phi}{v}) \tag{2}$$

The important physical results of these model is that the vacuum of the ground states are degenerate(two fold for the $\lambda\phi^4$ potential and infinity for the sin-Gordon model).

Consider small fluctuation around the vacuum, namely

$$\phi = v + \hat{\phi} \tag{3}$$

expands the potential of the scalar field in perturbative way, we find the mass($m^2 = \frac{\partial^2 V}{\partial \phi^2} >= 0$) of the corresponding scalar particle, due to fluctuation or excitation of the scalar field, as well as three and four point vertices, if the mass square $m^2 < 0$, these particle are called *tachyons*, which are not physical and would make the vacuum unstable.

The fact that the vacuum are degenerate, results in(by solving Euler-lagrange equation) these models posses highly non-trivial static solutions, that interpolate between the vacuum, which are called *solitons* or *kinks*, which are

$$\phi(x) = v\tanh\frac{1}{2}m(x - x_0), \quad \phi(x) = \frac{2v}{\pi}\arctan(e^{m(x-x_0)}) \tag{4}$$

for $\lambda\phi^4$ and sin-Gordon potentials, separately. The difference is the first one is a renormalizable one while the second is non-renormalizable. Where x_0 is position transition between two nearest vacuum take place.

The solution of solitons($\phi(x)$) are basicly located at the vacuum value(v), and can be approximated by a Heaviside function

$$\phi(x) = v\theta(x - x_0) \quad for \quad |x - x_0| \gg \Gamma \propto 1/m \tag{5}$$

which interpolate between two nearest vacuum, where the kinetic term and potential approach vanishes at spatial infinity($|x| \to \infty$). Moreover the solution has finite positive energy and it turns out to be antipropotional to the self couplings.

$$E \propto \frac{m^3}{\lambda} \tag{6}$$

which means the solitons are very heavy and massive in the perturbative limit. In the perturbation language, we are dealing with a theory describing elementary particles with massive gap due to light fluctuation of vacuum. There is an ambiguity due the the $\pm$ sign in front of the $\phi(x)$, which leads to two kind solutions, *solition*($\frac{\partial}{\partial x}\phi(x) > 0$) and *anti-soliton*($\frac{\partial}{\partial x}\phi(x) < 0$). The picture of the theory can be sequences of solitons and anti-solitons, in the "dilute gas" approximation, means the solutions of them only valid for widely separated objects($|x_{10} - x_{20}| \gg \frac{1}{m}$). The anti-slotions are anti-particle of the solitions like in a complex scalar field, and the vacuum can be distinguished by label it with the difference between the number of solitons and anti-solitions($\Delta n = n_+ - n_-$) for a potential with symmetry.

If we consider the chiral fermions in a soliton background, namely we couple the fermions to the solition by the usual Yukawa coupling.

$$\mathcal{L} = \bar{\phi}(\gamma^\mu \partial_\mu + g\phi(x))\phi \tag{7}$$

After spontaneous symmetry breaking of scalar field ϕ, it take the vacuum expectation value v every, the fermion also get a mass and become massive($m_\phi = gv$) and the Hamiltonian of fermions come in pairs $\pm E$ with $|E| \geq m$, "electrons" and "holenons" with positive and negative energy separately. The chiral symmetry of the fermions are broken due to mass term of fermions. But the introducing of the opposite vacuum solution will restore the symmetry, in other words, making the Yukawa couplings term chiral invariant again. Thus the chiral invariant vacuum state of fermions are a Z_2 degenerate vacuum, with ϕ and $-\phi$ corresponding to the same energy E for electrons or $-E$ for the holenons. We are also interested in the zero modes(with zero energy, and electron and holenons are the same particle since $\pm E = 0$) of the fermions in a solition background. It turns out to be that the only normalisable(non-divergent) zero mode state(wavefunctions) of fermions are spin up fermion (positive parity$P = +1$) localized at the transition position x_0 of the soliton and spin down fermion(negative parity $P = -1$) localized on the transition position of x_0 of the anti-solition.

For the concreteness, the wavefunctions of zero modes(Because they have eigen energy $E = 0$, they do not have effect on the energy whether the states are filled or not) of fermion are

$$\phi_\uparrow \sim \cosh\frac{m}{2}(x - x_0)^{-2\frac{m}{m_\psi}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \phi_\downarrow \sim \cosh\frac{m}{2}(x - x_0)^{-2\frac{m}{m_\psi}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8}$$

which are named Jackiw-Rebbi modes and they are both chiral eigen-state and related by a Z_2 discrete symmetry.

B. $1 + n(n \geq 2)$ solitons - Domain Wall or Membrane or 2-brane

The higher dimensional extension of solitons(kinks) are *domain walls*, which are transition regions between domains with different eigen-values Δn to label the different(degenerate) vacuums, they are related by a discrete symmetryx. The chiral fermions inside the domains are massive and on the domain walls they are massless, along the domain wall, we view them as lower dimensional Dirac fermions.

The first (2+1) domain walls are *vortices* or *strings*. For a complex scalar field(or two component real, where $\phi = \phi_1 + i\phi_2$),

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - \frac{\lambda}{2}(\phi^*\phi - \frac{v^2}{2})^2 \tag{9}$$

The *vacuum manifold*(moduli) is a circle S^1, with

$$\phi \to \frac{v}{\sqrt{2}} e^{i\varphi} \tag{10}$$

,

where φ is the polar angle in coordinate space. This complex scalar field theory in (2+1) dimension, with a $U(1)$ *global* invariance, characterize a vortex, and its energy $E = \int dx^2(\vec{\partial}\phi^*\vec{\partial}\phi + V(\phi, \phi^*))$ is *logarithmically divergent*. The approach to cure the divergence is by making the $U(1)$ global invariance a *local* one, namely it is necessary to introduce gauge field to cure the logarithmic divergent of the energy.

$$\partial\phi \to D_\mu\phi = (\partial_\mu - ieA_\mu)\phi \tag{11}$$

The covariant derivative$\vec{D}\phi$ kinetic term will give a good converge than the ordinary derivative$\vec{\partial}\phi$ kinetic term. It is clear that the logarithmic divergent is absorbed in the gauge field $\vec{A}$. Intuitively speaking, the gauge field has to be along in the direction of the phase φ. It turns out that

$$A_i \to -\frac{1}{e}\epsilon_{ij}\frac{x_j}{r^2} \tag{12}$$

and as expected it has only a φ component,

$$A_r \to 0, A_\varphi \to \frac{1}{er} \tag{13}$$

and it is a pure gauge asymptotically and the field strength vanishes,

$$\vec{A} \to \frac{1}{e}\vec{\partial}\varphi \qquad F_{ij} \to 0 \tag{14}$$

Thus the gauge filed $\vec{A}$ is a circular one and leads to a vortex with *quantized magnetic flux*,

$$\Phi = \int_S \vec{B} d\vec{\sigma} = \int_{C=\partial S} \vec{A} d\vec{x} \equiv n g_m = \frac{2\pi}{e} n = \frac{1}{e}(\varphi(r, \theta 2\pi) - \varphi(r, \theta)) \tag{15}$$

which is called *topological conservation law* for the original complex scalar field. The vortex carries a quantized magnetic *flux number* n, as the quantum number to label the different topological classes. The general total flux will be of the form $(n_1 + n_2 + n_3)\frac{2\pi}{e}$.

The choices for ϕ and A are solutions of the Euler-Lagrange equations asymptotically, the gauge field A runs into a singularity when we extend them naively towards the origin. Instead we can make an ansatz for ϕ and A and try to solve the problem.

The model we have discussed so far is a 2+1 dimensional Abelian Higgs Model with gauge strength term added,

$$\mathcal{L} = -D_\mu \phi^* D_\mu \phi - \frac{\lambda}{2}(\phi^* \phi - \frac{v^2}{2})^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{16}$$

then the energy of the model(static solution) is

$$\begin{aligned} E &= \int dx^2 [D_i \phi^* D_i \phi + \tfrac{1}{2} F_{12}^2 + \tfrac{\lambda}{2}(\phi^* \phi - \tfrac{1}{2} v^2)^2] \\ &= \int dx^2 [(\partial_i \phi)^2 + e^2 \phi^2 \vec{A}^2 + \tfrac{1}{2} B^2 + \tfrac{\lambda}{2}(\phi^* \phi - \tfrac{1}{2} v^2)^2] \\ &= \int dx^2 [(\partial_i \phi \pm e\epsilon_{ij} A_j \phi)^2 \pm e\phi^2 B + \tfrac{1}{2}(B \pm \sqrt{\lambda}(\phi^2 - \tfrac{v^2}{2}))^2 \mp \sqrt{\lambda}(\phi^2 - \tfrac{v^2}{2}) B] \end{aligned} \tag{17}$$

it turns out that before and after the spontaneously symmetry breaking, the massive scalar(Higgs) and the massive vector gauge boson have the mass

$$m_\phi = \sqrt{2} e \frac{v}{\sqrt{2}} = ev, \qquad m_A = \frac{\lambda}{2} v^2 \tag{18}$$

from the term $-\frac{\lambda}{2} v^2 |\phi|^2$ and $e^2 \frac{v^2}{2} \vec{A}^2$ separately. If we make a special choice, namely $\lambda = e^2$, the mass of the Higgs will be the same as that of vector gauge boson. After SSB, the energy will be of the form

$$E = \int d^2x [(\partial_i \phi \pm e\epsilon_{ij} A_j \phi)^2 \pm e\frac{v^2}{2} B + \frac{1}{2}(B \pm \sqrt{\lambda}(\phi^2 - \frac{v^2}{2}))^2] \geq e\frac{v^2}{2} |\int B d^2 x| \tag{19}$$

and has a lowest bound, which can be saturated with two equation of motions and it is

$$E \geq e\frac{v^2}{2} n \frac{2\pi}{e} = n\pi v^2 = n\frac{\pi m^2}{e^2} \tag{20}$$

The energy will be typically proportional to $\frac{m^2}{g^2}$ for a heavy topological objects in the perturbative viewpoint.

In a brief summary so far, it is the requirement of finite energy led to the configuration of vacuum state to fall into disjoint class, and the total configuration can be classified into different equivalent classes. Interpolating these classes have to include configuration with divergent energy.

To be general, a gauge symmetry group G is spontaneously broken down to one of its subgroup H, the vortex quantum flux number n form a group, $G_2 = \pi_1(G/G_1)$, called *first homotopy group*, is a mapping from S^1 into the coset G/G_1, which measures the non-contractibility of the coset. For example, an Abelian $U(1)$ gauge symmetry is SSB to identity $\mathbf{1}$, $\pi_1(S^1) = \pi_1(U(1)/\mathbf{1}) \equiv \pi_1(U(1)) = Z$. An non-Abelian $SU(2)$ gauge symmetry is SSB to a Z_2 symmetry, $\pi_1(SU(2)/Z_2) \equiv \pi_1(SO(3)) = Z_2$.

C. Physics of CS3 in (2+1)Dimensional space-time

In the lagrangian formalism,

$$\mathcal{S}_{CS} = c\int dx^3 \epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho = c\int dx^3 AdA = c\int dx^3 \vec{A}\cdot\vec{\nabla}\times\vec{A} = c\int dx^3 \vec{A}\cdot\vec{B} \tag{21}$$

On the other hand, the gauge kinetic term is

$$\mathcal{S}_{kin} = -\frac{1}{4}\int dx^3 F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int \vec{A}\cdot[\nabla^2\vec{A}^2 - \vec{\nabla}(\vec{\nabla}\cdot\vec{A})] \tag{22}$$

By variation of A, it is easy to get

$$\nabla^2\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = 2c\vec{B} \tag{23}$$

Imposing a curl gradient on both sides, we find

$$\nabla^2\vec{B} = 4c^2\vec{B} \tag{24}$$

Thus if $c \neq 0$, the $(2+1)$ spacetime photon will become massive.

The equation of motion is

$$\partial_\mu F^{\mu\nu} = j^\nu + J^\mu_{CS} \tag{25}$$

and the Chern-Simons current is

$$J^\mu_{CS} = 2c\epsilon^{\mu\nu\rho}\partial_\nu A_\rho = 2c\vec{\nabla}\times\vec{A} = 2c\vec{B} \tag{26}$$

The conservation law of Chern-Simons current is the magnetic field.

$$\partial_\mu J^\mu_{CS} = 0 \Longrightarrow \vec{\nabla}\cdot\vec{B} = 0 \tag{27}$$

Integrate out magnetic field of Eq.[21], according to the *topological quantization conditions*, we have

$$\mathcal{L}_{CS} = cng_m\int d\vec{x}\cdot\vec{A} = c\frac{2\pi\hbar n}{e}\int d\vec{x}\cdot\vec{A} \equiv q\int d\vec{x}\cdot\vec{A} \tag{28}$$

Where q is the effective charge of the fields, thus the CS coefficient is quantized as

$$c = \frac{qe}{2\pi\hbar n} \tag{29}$$

In another viewpoint, consider self-links of gauge field, after integrating out two closed line integral, or equivalent overlap of two flux from the corresponding magnetic field, we get the *topological quantization rule* as

$$c(\frac{2\pi\hbar n}{e})^2 = q\frac{2\pi\hbar n}{e} \tag{30}$$

This implies that if the effective charge is fractionalized, then it will lead to fractional statistics. In other words, if we have fermions satisfying fermion statistics, once turn on CS term, electron will become bosons satisfying boson statistics. We may call this effective fractional charge particle *Anyon*.

For example,
(a)if $q = \frac{e}{3}$, we will have a

$$\frac{2}{3}\pi n \tag{31}$$

phase difference.
(b)if $q = \frac{1}{2}e$, fermion, for $n \in$ odd.
(c)if $q = e$, boson.

D. (3+1)Dimensional non-Abelian gauge theory

We consider a general lagrangian of (3+1) dimensional non-Abelian gauge theory

$$\mathcal{L} = -\frac{1}{2}(D_\mu\phi^a)^2 - \frac{\lambda}{8}(\phi^{a2} - \frac{1}{2}v^2)^2 - \frac{1}{2}F^a_{\mu\nu}F^{\mu\nu a} \tag{32}$$

where both scalar field($O(3)$) and gauge field are $SU(2)$ invariant.

$$\phi = \phi^a\tau^a, \quad A_\mu = A^a_\mu\tau^a, \quad \tau^a = \frac{\sigma^a}{2} \tag{33}$$

and the non-Abelian covariant derivative and the field strength are

$$D_\mu\phi^a = \partial_\mu\phi^a + \epsilon^{abc}A_{\mu b}\phi_c, \quad F_{\mu\nu a} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu \tag{34}$$

The classical solution of the model turns out to be a "hedgehog", which has a zero in the origin point. In addition, it has a magnetic charge(so called 't Hooft and Polyakov monopole) inside.

The energy of the monopole is

$$E = \int d^3x[\tfrac{1}{2}(\vec{D}\phi^a) + \tfrac{\lambda}{8}(\phi^{a2} - \tfrac{1}{2}v^2)^2 + \tfrac{1}{2}(\vec{B}^a)^2] \tag{35}$$

$$= \int d^3x[\tfrac{1}{2}(\vec{D}\phi^a \pm \vec{B}^a)^2 + \tfrac{\lambda}{8}(\phi^{a2} - \tfrac{1}{2}v^2)^2] + \tfrac{2\pi}{e}v \geq \tfrac{4\pi}{e^2}m_V \tag{36}$$

Where $B^a_i = \frac{1}{2}\epsilon_{ijk}F^a_{jk}$, and the bound is saturated for vanishing potential, in the PS(Prasad and Sommerfield)limit $\lambda = 0$. After the $SU(2)$ gauge symmetry is spontaneously broken down to $U(1)$ by vacuum expectation value of an isovector field ϕ^a

The corresponding eigen-state of the monopole is so called BPS states with a mass

$$M_{mon} = \frac{4\pi}{e^2}m_V \tag{37}$$

which is anti-proportional to the gauge Higgs couplings as well as the gauge bosons mass.

E. The role of CS3 in 3+1 dimensional space-time physics and the Instanton

The action of 3+1 dimensional gauge field can be written as

$$S = -\int d^4x\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \tag{38}$$

we know that in order to make the energy non-divergent, we need to introduce gauge field to absorb the divergent and finally lead to the finite energy density, and a BPS bound,

$$S = -\int d^4x\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \geq \frac{8\pi^2}{g^2} \tag{39}$$

On the other hand, the action can also be simplified as

$$S = \int d^4x[-\tfrac{1}{8}((F^a_{\mu\nu})^2 + (\tilde{F}^a_{\mu\nu})^2)] = -\tfrac{1}{8}\int d^4x(F^a_{\mu\nu} - \tilde{F}^a_{\mu\nu}) + \tfrac{1}{4}\int d^4xF^a_{\mu\nu}\tilde{F}^{\mu\nu a} \tag{40}$$

$$= -\tfrac{1}{8}\int d^4x(F^a_{\mu\nu} - \tilde{F}^a_{\mu\nu}) + \tfrac{8\pi}{g^2}\partial_\mu K^\mu \tag{41}$$

where K^μ is the *gauge-variant* but *conserved* 2+1 Chern-Simons current with the form

$$K_\mu = \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}(A^a_\nu\partial_\rho A^a_\sigma + \frac{2g}{3}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma) \tag{42}$$

The surface integral on the S^3_∞ of Chern-Simons current term turns out to be

$$\frac{8\pi}{g^2}\int d^4xK_\mu = \frac{8\pi}{g^2}\int d^3xK_\perp = \frac{8\pi^2}{g^2} \tag{43}$$

which is just the minimal value from BPS bound.

Thus we can conclude that the BPS bound is saturated when the gauge field strength is selfdual,

$$F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu} \tag{44}$$

and the Bianchi identity is automatically fullfilled,

$$D_\mu F_{\mu\nu} = D_\mu \tilde{F}_{\mu\nu} = 0 \tag{45}$$

In the case, the lagrangian is a topological quantity proportional to $\frac{8\pi^2}{g^2}$, called Pontryagin index, which is independent of configuration of gauge fields.

Then as before, we consider the chiral fermions on these gauge field background including instanton. For the massless fermions, the vector and axial vector current are current at tree level, namely

$$\partial_\mu J^\mu = 0, \qquad \partial_\mu J^{\mu 5} = 0 \tag{46}$$

where

$$J_\mu = \bar{\psi}\gamma_\mu\psi = J_{\mu L} + J_{\mu R}, \quad J^5_\mu = \bar{\psi}\gamma_\mu\gamma^5\psi = J_{\mu R} - J_{\mu L} \tag{47}$$

At the loop level, the triangle fermion loop will contribute a nonzero part which is independent of regularization method. The ambiguity in γ^5 are removed by requiring that the Ward-Takahashi identity are valid for the vector gauge filed, so that the vector currents are still conserved at the loop level, but at the same time the axial vector currents are not.

$$\begin{aligned} &\partial_\mu < 0|J_\mu(x)|k_1 k_2 >= 0 \\ &\partial_\mu < 0|J^5_\mu(x)|k_1 k_2 >= -\tfrac{g^2}{16\pi^2} < 0|F^a_\mu \tilde{F}^a_\mu|k_1 k_2 > \end{aligned} \tag{48}$$

The non-conserved axial current is just the Adler-Bell-Jackiw anomaly,

From above, we already have $\int d^4x F^a_{\mu\nu}\tilde{F}^{\mu\nu a}$ from the instanton, thus the axial charge will be

$$Q^5 = \int d^3x J^5_0 = n_R - n_L \tag{49}$$

where n_R and n_L are respectively winding numbers of right and left handed zero mode fermions.

The difference $\Delta n = \frac{n_R - n_L}{2}$ means that the total flipped helicity of chiral fermions. In other words and to be concrete, the chiral fermion on the gauge field can feel the effect of instanton, and the instanton creates n_L right handed(LH) fermions and annihilates n_R reft handed(RH) fermions.

The instanton not only changes the winding number by one, but also creates a LH particle and annihilates a RH anti-particle. For the anti-instanton it is vice versa.

The instanton is related to θ vacuum and we can included into the effective lagrangian with $e^{\pm i\theta}$ for instanton and anti-instanton separately. To get the complete lagrangian at the same order, we have to add the topological surface term to get a complete one for gauge field,

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{g^2}{8\pi^2}\frac{i\theta}{4}F^a_{\mu\nu}\tilde{F}^{\mu\nu a} \tag{50}$$

Because of the couplings to $F\tilde{F}$, there are new sources of the gauge current. Varying the action with respect to A_μ, yields the general equation of motion,

$$\partial_\mu F^{\mu\nu} = j^\nu \quad \partial_\mu \tilde{F}^{\mu\nu} = k^\nu \tag{51}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{52}$$

with current

$$j^\nu = \frac{g^2}{8\pi^2}\frac{i}{4}\partial^\mu(\theta\tilde{F}^{\nu\mu a}) \tag{53}$$

and the four vector potential

$$A^\mu = (A^0, \vec{A}) \tag{54}$$

written in non-covariant form in terms of electric and magnetic sources, the currents become $j^\mu = (\rho, \vec{j})$, $k^\mu = (\sigma, \vec{k})$, from the two of Maxwell's equation

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0 \tag{55}$$

follows from the Bianchi identity $dF = 0$.

In the presence of magnetic source, e.g., a point magnetic monopole of unit charge, the magnetic filed $\vec{B}$ is still given by the curl of vector potential $\vec{\nabla} \times \vec{A}$, but the diverse is nonzero $\vec{\nabla} \cdot \vec{B} \neq 0$.

the equations above are promoted to

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu \tilde{F}^{\rho\sigma} = k^\nu \tag{56}$$

thus

$$\frac{1}{2}\epsilon^{0\mu\rho\sigma}\partial_\mu \tilde{F}^{\rho\sigma} = -\delta^3(\vec{x}) \tag{57}$$

One can impose the Bianchi identity $\partial_\mu \tilde{F}^{\mu\nu} = 0$ in the electric theory by introducing a Lagrange multiplier - a dual gauge vector field $A_{D\mu}$ which couples to the monopole,

$$\mathcal{L}_D = \frac{1}{8\pi} A_{D\mu}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = \frac{1}{8\pi}\tilde{F}_D^{\rho\sigma} F_{\rho\sigma} = \frac{1}{16\pi}\mathrm{Im}(F_D^{\rho\sigma} + i\tilde{F}_D^{\rho\sigma})(F^{\rho\sigma} + i\tilde{F}^{\rho\sigma}) \tag{58}$$

we have

$$\begin{aligned} \rho &\sim \theta\vec{\nabla}\cdot\vec{B} + \vec{B}\cdot\vec{\nabla}\theta \\ \vec{j} &\sim \dot{\theta}\vec{B} + \vec{\nabla}\theta \times \vec{E} \end{aligned} \tag{59}$$

The first term in ρ is responsible for the Witten effect-monopoles in the electric field or nono-zero CP angle carry electric charge. The second term in $\vec{j}$ give a Hall-like contribution to the current which is perpendicular to the applied electric field.

in the canonical normalization basis, it is

$$\mathcal{L} = -\frac{1}{4g^2}F^a_{\mu\nu}F^{\mu\nu a} + \frac{i\theta}{8\pi^2}\frac{1}{4}F^a_{\mu\nu}\tilde{F}^{\mu\nu a} \tag{60}$$

Thus for a pure gauge theory, we have two nature constants, namely gauge coupling g and θ vacuum, both of which are in principle observable. We can define

$$\tau = \frac{4\pi}{g^2} + \frac{i\theta}{2\pi} \tag{61}$$

which is the holomorphic variable in supersymmetry theories(SUSY). It is symmetric under θ vacuum translation by integer times of 2π, namely $\tau \to \tau + 1$, which is the so called *T duality*

The Lagrangian with the holographic gauge coupling can be written as

$$\mathcal{L} = \frac{1}{16\pi}\mathrm{Im}\tau(F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}\tilde{F}^{\mu\nu}) = \frac{1}{32\pi}\mathrm{Im}\tau(F_{\mu\nu} + i\tilde{F}_{\mu\nu})^2 \tag{62}$$

combing above Lagrangian and that coupled to dual magnetic gauge vector field $\mathcal{L}_D$, and integrate out the original electric fields $F_{\mu\nu} + i\tilde{F}_{\mu\nu}$, gives the EOM for the dual electomagnetic fields which couples to the monopole

$$F_{\mu\nu} + i\tilde{F}_{\mu\nu} = -\frac{1}{\tau}(F_D^{\mu\nu} + i\tilde{F}_D^{\mu\nu}) \tag{63}$$

substituting above EOM into the Lagrangian, we get one for the pure dual electromagnetic fields

$$\mathcal{L}_D = \frac{1}{32\pi}\mathrm{Im}(-\frac{1}{\tau})(F_D^{\mu\nu} + i\tilde{F}_D^{\mu\nu})^2 = \frac{1}{32\pi}\mathrm{Im}\tau_D(F_D^{\mu\nu} + i\tilde{F}_D^{\mu\nu})^2 \tag{64}$$

where

$$\tau_D = -\frac{1}{\tau} \tag{65}$$

is the holomorphic gauge coupling in the dual magnetic theory, means the couplings in the dual electric and magnetic theories are inverse proportional to each other, when electric theory is strongly coupled in the non-perturbative region, as a result the perturbative approach fails, meanwhile the corresponding dual magnetic theory arise, which is a weakly coupled gauge theory and thus an appropriate descriptions for the same theory in perturbative sense.

Note that $F^{\mu\nu}F^{\mu\nu} = -\tilde{F}^{\mu\nu}\tilde{F}^{\mu\nu} = -2(\vec{E}^2 - \vec{B}^2)$, the generalize EOM are invariant under *electric-magnetic duality*(The quantitative definition is the so called *S duality*, namely $\tau \to \tau_D = -\frac{1}{\tau}$),

$$F^{\mu\nu} \to \tilde{F}^{\mu\nu}, \quad \tilde{F}^{\mu\nu} \to -F^{\mu\nu}, \quad j^\mu \to k^\mu, \quad k^\mu \to -j^\mu \tag{66}$$

which correspond to interchanging the electric and magnetic fields as

$$\vec{E} \to \vec{B} \quad \vec{B} \to -\vec{E} \tag{67}$$

The lagrangian can be simplified as

$$\mathcal{L} = \tfrac{1}{2}(\vec{E}^a - \vec{B}^a) - \tfrac{\theta g^2}{8\pi^2}\vec{E}^a \cdot \vec{B}^a \tag{68}$$

$$= \tfrac{1}{2}(\vec{E}^a - \tfrac{\theta g^2}{8\pi^2}\vec{B}^a)^2 - \tfrac{1}{2}(1 + \tfrac{\theta^2 g^4}{(8\pi^2)^2})(\vec{B}^a)^2 = \tfrac{1}{2}(\vec{E}^{a'})^2 - \tfrac{1}{2}(\vec{B}^{a'})^2 \tag{69}$$

Integrate both fields, we find,

$$q_e^{'} = q_e - \frac{\theta g^2}{8\pi^2} g_m \tag{70}$$

If we integrate out $E^{'}$ field around a monopole B, we find

$$q_e^{'} = 0, \quad g_m = \frac{4\pi}{g}, \quad q_e = \frac{\theta g}{2\pi} \tag{71}$$

Which means the **monopole behaves like that it has a *fractional electric charge***,

On the other hand, we already know that by binding to fermions, the monopole could get an integer electric charge, thus every electric and magnetic charge become periodic in θ. Because the monopole behaves like it has a fractional charge, the *Dirac quantization condition* between electric and magnetic charge should be more democratic

$$q_e^1 g_m^2 - q_e^2 g_m^1 = 2\pi n_{12}, \quad n_{12} \in Z \tag{72}$$

which means a quantum theory can have two dyons(dyons are particles that carry both electric and magnetic charge) with charge (g_1, e_1) and (g_2, e_2) only if above topological quantization conditions are satisfied.

Now consider a monopole of charges $(\frac{2\pi}{e}, q)$, if CP is invariant, we have another monopole with charges $(\frac{2\pi}{e}, -q)$, since electric charger is odd under CP but the magnetic charge is even. The requirement of the generalized Dirac quantization condition shown above require that

$$q = n_{12} e \quad \text{or} \quad q = (n + \frac{1}{2})e \tag{73}$$

means the monopole must have integer or half-integer charges. Moreover, if monopole of integer charge exist, then monopoles of half-integer charge do not and vice-versa. The observable weakly violated CP in nature implies that if monopole exist, they will have charges that are almost but not quite integers and the deviation from integral charge would be proportional to the strength of CP violation.

The interesting way to introduce the CP violation into the theory is to consider a non-zero value of the θ vacuum angle by adding to the Lagrangian an additional CP violating interaction term to the (3+1) $SU(2)$ non-Abelian model which provide a so called 't Hooft Polyakov magnetic monopole after spontaneously symmetry breaking with mass $m_V = \frac{4\pi}{e^2} m_V$, as shown in the Eq.(32),

$$\mathcal{L}_{CP} = \frac{\theta e^2}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{74}$$

when $\theta \neq 0$, CP is not conserved.

It turns out that(See Witten's 1979 paper "Dyons of charge $\frac{e\theta}{2\pi}$") the charge of the monopole becomes

$$q = ne - \frac{\theta e}{2\pi} \equiv (n+\delta)e \quad \delta = -\frac{\theta}{2\pi} \tag{75}$$

means the magnetic monopoles electric charge depend on θ and are not integral if θ is non-zero, in particular, if θ is not zero, in other word, CP is violated, there does not exist an electrically neutral magnetic monopole(q can never be 0). Assuming that θ vacuum angle is the only source of the CP violation(note: CP violation can also origins from other source in particle physics), the monopole charge $q = (n+\delta)e$ is exact without higher order corrections.

Note: the θ vacuum dependence of the monopole charge here do not refer to instantons[3]. If monopole is absent, no classically allowed motions with non-zero $F\tilde{F}$ term appear in the theory, the θ vacuum dependence appear as a tunneling effect of order $e^{-\frac{4\pi}{e^2}}$ contributed from instantons. However, in the presence of monopole, there are classically allowed motions for dyons, with non-zero $F\tilde{F}$ term appear. This is clear in our electric-magnetic duality analysis before. Consequently, the θ dependence in the monopole sector has nothing to do with instantons and is of leading order rather than order of $e^{-\frac{4\pi}{e^2}}$

F. Topological mass generation mechanism

The mass generation mechanism can be understood in terms of topological interactions, without requiring the detail knowledge of the underlying dynamics. It is well known that in 1+1 dimensional QED, the photon will become massive due to topological interactions.

1. 1+1 Dimensional Schwinger's model

The model is a Abelian QED, with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\gamma_\mu(\partial_\mu + ieA_\mu)\psi \tag{76}$$

the QED interaction is

$$\mathcal{L}_{int} = -eA_\mu\bar{\psi}\gamma^\mu\psi \equiv -eA_\mu J^\mu \tag{77}$$

We already know the anomalous divergent is

$$\partial_\mu J^{\mu 5} = -\frac{e}{2\pi}\epsilon^{\mu nu}F_{\mu\nu} = \frac{e}{\pi}F \tag{78}$$

where F is a pseudo scalar. where the dual curvature is just the pseudo scalar.

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu} = -F \tag{79}$$

The axial vector current J^5_μ is dual to the vector current $J^5_\mu = \epsilon_{\mu\nu}J^\nu$, in the 2D Γ matrix, the duality is the identity

$$\gamma_\mu\gamma^5 = \epsilon_{\mu\nu}\gamma^\nu \tag{80}$$

From the equation of motion,

$$\partial_\mu F^{\mu\nu} = eJ^\nu \tag{81}$$

we get

$$\epsilon_{\alpha\nu}\epsilon^{\mu\nu}\partial_\mu F = e\epsilon_{\alpha\nu}J^\nu \tag{82}$$

or $-g^\nu_\mu\partial_\nu F = eJ^5_\mu$, namely $\partial_\mu F = -eJ^5_\alpha$, thus we find

$$\partial^2 F + \frac{e^2}{\pi}F = 0 \tag{83}$$

(Note: the explicit step by step derivation above is not necessary but helpful in dealing with 4D theory.)

The Chern-Simons current is

$$K_2^\mu \equiv \epsilon^{\mu\nu} A_\nu \tag{84}$$

and the Equation of the motion of the Chern current is

$$\partial_\mu K_2^\mu = \epsilon^{\mu\nu}\partial_\mu A_\nu = \frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu} = \tilde{F}_{\mu\nu} = -F. \tag{85}$$

The Lagrangian become

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - eJ^\mu A_\mu = \frac{1}{2}F^2 - eA_\mu\epsilon^{\mu\nu}J_\nu^5 = \frac{1}{2}(\tilde{F}_{\mu\nu})^2 + eK_2^\mu J_\mu^5 \tag{86}$$

2. 3+1 Dimensional non-Abelian model with topological mass generation

The Chern-Simons currents is

$$K_4^\mu \equiv 2\epsilon^{\mu\nu\rho\sigma}(A_\nu^a\partial_\rho A_\sigma^a + \frac{1}{3}f^{abc}A_\nu^a A_\rho^b A_\sigma^c) \tag{87}$$

and the dual field is

$$\tilde{F}^{\mu\nu a} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}^a \tag{88}$$

the equation of motion is

$$\partial_\mu K_4^\mu = \tilde{F}^{\mu\nu a}F_{\mu\nu}^a \tag{89}$$

Variation of the Chern-Simons current gives

$$\begin{aligned}\delta K_4^\mu &= 2\epsilon^{\alpha\mu\nu\omega}[(\partial_\nu A_\omega^a)\delta A_\mu^a + f^{abc}\delta A_\mu^a A_\nu^b A_\omega^c] \\ &= 2\epsilon^{\alpha\mu\nu\omega}[F_{\nu\omega}^a\delta A_\mu^a - \partial_\nu[(\delta A_\omega^a)A_\mu^a]] = 4F^{\alpha\mu a}\delta A_\mu^a + 2\epsilon^{\alpha\omega\nu\mu}\partial_\nu[(\delta A_\omega^a)A_\mu^a]\end{aligned} \tag{90}$$

The duality identity corresponds to that of 2D is

$$\epsilon^{\mu\nu\omega\alpha}\gamma_\alpha\gamma^5 = g^{\mu\nu}\gamma^\omega - g^{\mu\omega}\gamma^\nu + g^{\nu\omega}\gamma^\mu - \gamma^\mu\gamma^\nu\gamma^\omega \tag{91}$$

The axial vector current is anomalous as expected,

$$\partial^\mu J_\mu^5 = -\kappa\tilde{F}^{\mu\nu a}F_{\mu\nu}^a \tag{92}$$

where κ is a positive numerical coupling constant.
By taking the divergence of above equation, we get

$$J_\mu^5 = -\frac{\kappa}{\partial^2}\partial_\mu\tilde{F}^{\mu\nu a}F_{\mu\nu}^a \tag{93}$$

The Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2}(\tilde{F}^{\mu\nu a}F_{\mu\nu}^a)^2 + \Lambda^2 K_4^\mu J_\mu^5 \tag{94}$$

where Λ^2 carries a mass-squared dimension, since $(\tilde{F}^{\mu\nu a}F_{\mu\nu}^a)^2$ is dimension eight operator and $K_4^\mu J_\mu^5$ is a dimension six one.
Variation of the Lagrangian with respect to A_μ^a gives the equation of motion(EOM),

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta A_\alpha^a} &= \tilde{F}^{\mu\nu a}F_{\mu\nu}^a\frac{\partial[\tilde{F}^{\mu\nu a}F_{\mu\nu}^a]}{\partial A_\alpha^a} + \Lambda^2\frac{\partial K_4^\mu}{\partial A_\alpha^a}J_\mu^5 \\ &= 4[\tilde{F}^{\mu\nu a}F_{\mu\nu}^a]\partial_\beta\tilde{F}^{\beta\alpha a} + 4\Lambda^2 J^{5\alpha} + 2\partial_\nu[\Lambda^2 J_\mu^5]\epsilon^{\alpha\nu\omega\mu}A_\omega^a \\ &= -4\partial_\beta[\tilde{F}^{\mu\nu a}F_{\mu\nu}^a]\tilde{F}^{\beta\alpha a} + 4\Lambda^2 J^{5\alpha} + 2\partial_\nu[\Lambda^2 J_\mu^5]\epsilon^{\alpha\nu\omega\mu}A_\omega^a = 0\end{aligned} \tag{95}$$

If Λ^2 is a physical energy scale like QCD by dimensional transmutation, and thus a constants, the second part of the EOM can be simplified by exchange index α and ω as

$$\Lambda^2(\partial_\nu J^5_\mu - \partial_\mu J^5_\nu)\epsilon^{\alpha\nu\omega\mu} A^a_\omega \tag{96}$$

thus is zero.

By using the identity $\tilde{F}^{\mu\nu} F_{\mu\rho} = \frac{1}{4}\delta^\nu_\rho \tilde{F}^{\mu\nu a} F^a_{\mu\nu}$(by assuming $\tilde{F}^{\mu\nu} F_{\mu\rho} = A\delta^\nu_\rho[\tilde{F}^{\mu\nu a} F^a_{\mu\nu}]$, by taking trace of both sides to get $A = \frac{1}{4}$), we get the EOM

$$-\partial_\alpha[\tilde{F}^{\mu\nu a} F^a_{\mu\nu}] + \Lambda^2 J^5_\alpha = 0 \tag{97}$$

Take the divergence on both sides, we arrive

$$-\partial^2[\tilde{F}^{\mu\nu a} F^a_{\mu\nu}] + \Lambda^2 \partial^\alpha J^5_\alpha = 0 \tag{98}$$

Finally, we obtain the EOM in the formalism with the anomalous divergence of J^5_α,

$$\partial^2[\tilde{F}^{\mu\nu a} F^a_{\mu\nu}] + \kappa\Lambda^2[\tilde{F}^{\mu\nu a} F^a_{\mu\nu}] = 0 \tag{99}$$

which means $\tilde{F}^{\mu\nu a} F^a_{\mu\nu}$ propagates as a free massive field. In the another viewpoint, the pseudo-scalar, or the condensation of gauge field $< F^{\mu\nu a} F^a_{\mu\nu} >$ has obtains a mass

$$m^2 = \kappa\Lambda^2 \tag{100}$$

At the moment, we can make some comments here: On one hand, the anomaly from the axial current provides as a source of mass generation mechanism(i.e., if there is no anomaly $\kappa = 0$, then $m = 0$), for the $(3+1)$ space-time, the coupling κ is automatically generated from the triangle loop diagram of fermions. On the other hand, the mass is proportional to the physical dynamical scale Λ. The explanation of mass generation mechanism might attributes to where and how the dynamical energy scale Λ arise.

II. HOLOMORPHIC GAUGE COUPLINGS AND NSVZ β FUNCTIONS

In the following, we extend our discussion in ordinary space to the super space. Supersymmetry is asymmetry which relates bosons and fermions. Because on one hand, fermion loops contribute minus sign comparing with gauge bosons in Coleman Weinberg potential, on the other hand, bosons always appear with fermion partners, thus ultraviolet divergence are canceled between bosonic and fermionic loops. Thus the larger hierachy and fine-tuning problems of QFT are solved under the frame of low scale SUSY. The important question left is how SUSY is broken. SUSY gauge theories share features like confinement and chiral symmetry breaking as in QCD and served as one of the ideal laboratories for constructing strongly coupled gauge theories those could give important clues in dealing with strongly coupled non-SUSY QCD.

We begins from N=1 SUSY Lagrangian with $SU(N)$ gauge theory with N_f flavors.

$$\mathcal{L} = \int d^4\theta \Phi^\dagger e^{-2V}\Phi + \frac{1}{8\pi}\mathrm{Im}(\tau \mathrm{Tr}\int d^2\theta \mathcal{W}^\alpha W_\alpha) + (\int d^2\theta W(\Phi) + h.c.) \tag{101}$$

with the scalar chiral superfield Φ, vector superfield V, vector field strength chiral superfield W^α and holomorphic superpotential term $W(\Phi)$, where

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \tfrac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \tfrac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) \tag{102}$$

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta^2\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x) \tag{103}$$

in the Wess-Zumion gauge.

$$\mathcal{W}_\alpha = (-i\lambda^a_\alpha + \theta_\alpha D^a - (\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F^a_{\mu\nu} + \theta^2(\sigma^\mu D_\mu\bar{\lambda}^a)_\alpha)T^a \tag{104}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$ is the ordinary non-Abelian gauge field strength, and gauge covariant kinematic term for gaugino $D_\mu\bar{\lambda}^a = \partial_\mu\bar{\lambda}^a + f^{abc}A^b_\mu\bar{\lambda}^c$. Finally, the holomorphic superpotential terms(after Tailor expansion in superspace)

$$W(\Phi_i) = W(\phi_i) + \frac{\partial W}{\partial\phi_i}\sqrt{2}\theta\psi_i + \theta^2(\frac{\partial W}{\partial\phi_i}F_i - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j) \tag{105}$$

(Note: $R[\psi] = R[\phi] - 1$, $R[F] = R[\phi] - 2$, $R[\theta] = 1$,$R[W] = 2$, $R[\lambda] = 1$)

After some algebra, the total Lagrangian of N=1 SUSY can be expressed as

$$\begin{aligned}\mathcal{L} = &(D_\mu\varphi)^\dagger D^\mu\varphi - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi - D^a\varphi^\dagger T^a\varphi - i\sqrt{2}\varphi^\dagger T^a\lambda^a\psi + i\sqrt{2}\bar{\psi}T^a\lambda^a\varphi \\ &+F_i^\dagger F_i - \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \tfrac{g^2}{8\pi^2}\tfrac{i\theta}{4}F^a_{\mu\nu}\tilde{F}^{\mu\nu a} - \tfrac{i}{g^2}\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + \tfrac{1}{2g^2}D^aD^a \\ &+\tfrac{\partial W}{\partial\varphi_i}F_i + \tfrac{\partial\bar{W}}{\partial\varphi_i^\dagger}F_i^\dagger - \tfrac{1}{2}\tfrac{\partial^2 W}{\partial\varphi_i\partial\varphi_j}\psi_i\psi_j - \tfrac{1}{2}\tfrac{\partial^2\bar{W}}{\partial\varphi_i^\dagger\partial\varphi_j^{dag}}\bar{\psi}_i\bar{\psi}_j\end{aligned} \tag{106}$$

It is straightforward to prove that the Lagrangian is invariant under the SUSY transformation, and in addition is closed,

$$\begin{aligned}\delta A^a_\mu &= -\tfrac{1}{\sqrt{2}}(\epsilon^\dagger\bar{\sigma}^\mu\lambda^a + \lambda^{a\dagger}\bar{\sigma}_\mu\epsilon) \\ \delta\lambda_\alpha &= -\tfrac{i}{2\sqrt{2}}(\sigma^\mu\bar{\sigma}_\nu\epsilon)_\alpha F_{\mu\nu} + \tfrac{1}{\sqrt{2}}\epsilon_\alpha D \\ \delta\lambda_{\cdot\alpha} &= \tfrac{i}{2\sqrt{2}}(\epsilon^\dagger\bar{\sigma}^\nu\sigma^\mu)_{\cdot\alpha}F_{\mu\nu} + \tfrac{1}{\sqrt{2}}\epsilon^\dagger_\alpha D \\ \delta D &= -\tfrac{1}{\sqrt{2}}[\epsilon^\dagger\bar{\sigma}^\mu D_\mu\lambda - D_\mu\lambda^\dagger\bar{\sigma}^\mu\epsilon]\end{aligned} \tag{107}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$, and $D_\mu\lambda^a = \partial_\mu\lambda^a - gf^{abc}A^b_\mu\lambda^c$.

Eliminate the auxiliary F and D filed by using EOM

$$F_i^\dagger = -\frac{\partial W}{\partial\varphi_i} \quad D^a = g^2\varphi^\dagger T^a\varphi \tag{108}$$

the total Lagrangian can be simplified as

$$\begin{aligned}\mathcal{L} = &(D_\mu\varphi)^\dagger D^\mu\varphi - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi - D^a\varphi^\dagger T^a\varphi - i\sqrt{2}\varphi^\dagger T^a\lambda^a\psi + i\sqrt{2}\bar{\psi}T^a\lambda^a\varphi \\ &-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \tfrac{g^2}{8\pi^2}\tfrac{i\theta}{4}F^a_{\mu\nu}\tilde{F}^{\mu\nu a} - \tfrac{i}{g^2}\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a - V - \tfrac{1}{2}\tfrac{\partial^2 W}{\partial\varphi_i\partial\varphi_j}\psi_i\psi_j - \tfrac{1}{2}\tfrac{\partial^2\bar{W}}{\partial\varphi_i^\dagger\partial\varphi_j^{dag}}\bar{\psi}_i\bar{\psi}_j\end{aligned} \tag{109}$$

where the scalar potential consists of F terms and D terms

$$V = \sum_i F_i^\dagger F_i + \frac{1}{2g^2}D^aD^a = \sum_i |\frac{\partial W}{\partial\varphi_i}|^2 + \frac{1}{2}g^2(\varphi^\dagger T^a\varphi)^2 \tag{110}$$

For N=1 SUSY, we have an additional $U(1)_R$ global R symmetry(Hidden in non-SUSY theory). By requiring chiral and anti-chiral superfields have zero R charge, scalar will have zero R charge, but Weyl fermions with a -1 hypercharge, and gaugino a +1 R charge. R symmetry is behaviors like a chiral symmetry, and thus it is anomalous in the quantum theory, labeled by $U(1)^{'}_R$, combining which with the ordinary anomalous $U(1)_A$ symmetry, it is possible to construct a new anomaly free $U(1)_R$ symmetry, with the R charge

$$R = R^{'} + \frac{N_f - N}{N_f}A \tag{111}$$

TABLE I: Electric theory

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Φ^f	□	□	1	1	$1-\frac{N}{N_f}$
$\tilde{\Phi}_f$	□	1	□	-1	$1-\frac{N}{N_f}$
$\Phi^f\tilde{\Phi}_g$	1	□	□	0	2(1-$\frac{N}{N_f}$)

For the discussion at the moment, we focus on the terms refers to holomorphic gauge couplings

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$$

. When the gauge couplings g becomes stronger, the non-perturbative contribution from the θ vacuum(multi instantons) will contribute effect.

These comes from second term of Eq.(135), which relates to vector field strength chiral superfield W^α. After some algebra, by and finally by using the identity

$$\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\rho\sigma}] = -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}) \tag{112}$$

(with Wess & Bagger 's notation $\epsilon_{0123} = -1$), and neglecting the gaugino term and D terms, we arrive at the pure gauge interactions with instanton term before, namely

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{g^2}{8\pi^2}\frac{i\theta}{4}F^a_{\mu\nu}\tilde{F}^{\mu\nu a} \tag{113}$$

With the holomorphic gauge coupling τ running in the renormaliation group(RG), the one loop β function is

$$\beta(g) = \frac{\partial g}{\partial \log\mu} = -\frac{b_0}{16\pi^2}g^3 \tag{114}$$

with renormalization coefficient $b_0 = 3C_2(G_c) - \sum_f C(r_f)$ determine by the representation of chiral superfields $r_f \in G_c$. Where $C_2(r)1 = (T^aT^a)_r$ is quartic Casimir operator and $C(r)\delta^{ab} = \mathrm{Tr}_f(T^aT^b)$ and adjoint representation G_c, i.e., for SUSY QCD, we get

$$b_0 = 3N - N_f \tag{115}$$

(note: $C(r_f) = \frac{1}{2}$ for $r = \Box$ or $\Box$, and N for $r \in G_c$, $C_2(N) = \frac{N^2-1}{2N}$, $C_2(G_c) = N$ for $SU(N)$)

The solution for the running coupling is

$$\frac{1}{g^2(\mu)} = -\frac{b_0}{8\pi}\log\frac{\Lambda}{\mu} \tag{116}$$

where $\Lambda > 0$ is the intrinsic non-perturbative energy scale of the non-Abelian gauge theory that enters trhrough dimensional transmutation.

The holomorphic gauge couplings τ at the one loop level, can be rewritten as

$$\tau_{1-loop} = \frac{1}{2\pi i}\log[(\frac{\Lambda}{\mu})^{b_0}e^{i\theta}] \tag{117}$$

For $N_f \leq 3N$, given the gauge couplings$g$ at some energy scale μ, it is obviously that the gauge couplings becomes larger as the energy scale becomes smaller. Thus it is nature to define the non-Perturbative scale of the theory as

$$\Lambda = \mu e^{\frac{2\pi i\tau - i\theta}{b_0}} = \mu e^{-\frac{8\pi^2}{g^2 b_0}} \tag{118}$$

with μ a large energy scale where the gauge couplings constant g is given, i.e., for SUSY QCD, the non-perturbative scale is determined by above equation if $N_f < 3N$.

On the other hand, we know the two loop running of the gauge coupling for $N = 1$ SUSY QCD with N_f flavors is known,

$$\beta(g)_2 = -\frac{1}{16\pi^2}(3N - N_f)g^3 + \frac{1}{64\pi^4}(4NN_f - 6N^2 - \frac{2N_f}{N})g^5 \tag{119}$$

it is expected the one loop contribution(the first term) is dominate at high energy since $g \ll 1$. However, when the theory goes flow to low energy, the gauge couplings become stronger, and at some critical point$g(\mu)$, the two loop term might become comparable to the one loop term and the β function gives zero. The conditions are equivalent to

$$\frac{3N}{2} < N_f < 3N \tag{120}$$

the first in-equivalent identity make the one loop term negative, while the second in-equivalent identity make the two loop term positive. This analysis suggests a conformal fixed point.

The interesting thing is not the running of gauge couplings, but the running of the holomorphic gauge couplingsτ. With the properties that $\tau(\mu) = \frac{4\pi i}{g}(\mu) + \frac{\theta}{2\pi}$ is holomorphic and periodic in the Yang-Mills phase θ, it is straightforward to prove the so called nonrenormalization theorem that the holomorphic gauge coupling is exhausted at one-loop, therefore not renormalized by higher loops in perturbation theory and the one loop logarithmic running is exact. On the other hand, the couplings in the tree level superpotential are not renormalized at any order in perturbation theory, since the more general holomorphic dynamical effective superpotential in the perturbative limit($g \to 0$) is the same as the Wess-Zumino cubic tree level superpotential for a renormalizable theories.

$$W_{eff} = W_{tree} = \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3 \tag{121}$$

Note:The tree level superpotential has mass dimension 3 for renormalizable theories, since $\int d^2\theta W$ contains product of component fields of mass dimension 4 when W has cubic interaction in terms of scalar chiral superfields.

In the following, consider two left handed chiral super fields $(\Phi^f, \tilde{\Phi}_g)$ belong to the fundamental($\Box$) and anti-fundamental representation ($\bar{\Box}$) of SU(N), separately. For the right handed ones, just denote $(\bar{\Phi}^f, \bar{\tilde{\Phi}}_g)$

$$\mathcal{L} = \int d^4\theta(\Phi_f^\dagger e^{-2V}\Phi_f + \tilde{\Phi}_g e^{-2V}\tilde{\Phi}^{g\dagger}) + \frac{1}{8\pi}\text{Im}(\tau\text{Tr}\int d^2\theta\mathcal{W}^\alpha W_\alpha) + (\int d^2\theta W(\Phi,\bar{\Phi}) + h.c.) \tag{122}$$

Once the holomorphic gauge couplings is RG running from cutoff μ to $\mu^{'}$, namely

$$\tau(\mu^{'}) = \tau(\mu) + \frac{ib_0}{2\pi}\log\frac{\mu}{\mu^{'}} \tag{123}$$

with running coefficient $b_0 = 3C_2(G_c) - \sum_f C(r_{\Phi,\bar{\Phi}})$.

the chiral superfields $(\Phi,\bar{\Phi})$ are renormalized, and contribute in the nonholomorphic Kahler kinetic term

$$\mathcal{L} = \int d^4\theta Z_{\Phi,\bar{\Phi}}(\mu,\mu^{'})(\Phi_f^\dagger e^{-2V}\Phi_f + \tilde{\Phi}_g e^{-2V}\tilde{\Phi}^{g\dagger}) + \frac{1}{8\pi}\text{Im}(\tau(\mu^{'})\text{Tr}\int d^2\theta\mathcal{W}^\alpha W_\alpha) \tag{124}$$

Because the wavefunction renormalization coming from the Kahler kinetic terms in the Lagrangian, the running of the physical gauge couplings g is not holomorphic.

Canonically normalize the chiral superfield just behaviors like the chiral gauge transformation of $U(1)_A$ symmetry for matter, contributes to measure of chiral superfields in the path integral $\mathcal{D}\Phi\mathcal{D}\Phi^\dagger\mathcal{D}\bar{\Phi}\mathcal{D}\bar{\Phi}^\dagger$ with an additional phase

$$e^{\int d^4x\frac{1}{16\pi^2}\text{Tr}[iC(r_{\Phi,\bar{\Phi}})i\log Z_{\Phi,\bar{\Phi}}\tau\text{Tr}\int d^2\theta\mathcal{W}^a\mathcal{W}_a]} \tag{125}$$

From which, we get the physical running of the gauge couplings coefficients, after applying differential $\frac{\partial}{\partial\log\mu}$, we arrive the physical Novikow-Shifman-Vainshtein-Zakarov(NSVZ) one loop beta functions, which is the exact result for all loops.

$$\beta(g) = \frac{\partial g}{\partial\log\mu} = \frac{3C_2(G_c) - \sum_f C(r_{\Phi,\bar{\Phi}})(1-\gamma_{\Phi,\bar{\Phi}})}{1 - C_2(G_c)\frac{g^2}{8\pi^2}}\frac{g^3}{16\pi^2} \tag{126}$$

where

$$\gamma_{\Phi,\bar{\Phi}} = \frac{\partial\log Z_{\Phi,\bar{\Phi}}}{\partial\log\mu} \tag{127}$$

is the anomalous mass dimension of chiral superfields origins from wave function renormalization of matter belong the representation r_f(the mass dimension of the field operator rescale $d[\Phi] \to d[\sqrt{Z}\Phi] = 1+\frac{\gamma}{2}$), and the non-zero gauge couplings term in the denominator of above equation comes from the wavefunction renormalization of gaugino from kinetical basis to canoical basis($\frac{1}{g^2}F^2 \to F^2$).

Take $SU(N)$ SQCD as an example, we have

$$\beta(g) = \frac{3N - N_f(1-\gamma)}{1 - N\frac{g^2}{8\pi^2}}\frac{g^3}{16\pi^2} \tag{128}$$

in terms of 't Hooft coupling $g_t = g\sqrt{N}$, it becomes

$$\beta(g_t) = \frac{3N - N_f(1-\gamma)}{1 - \frac{g_t^2}{8\pi^2}}\frac{g_t^2}{8\pi^2 N} \tag{129}$$

From which, we get the critical values of anomalous dimension

$$\gamma_c = 1 - \frac{3N}{N_f} \tag{130}$$

where the β function always vanish for γ_c. As $N_f = 3N$, $\gamma_c = 0$ and as N_f decrease, γ_c become negative. Thus we find the mass dimension of the rescaled scalar chiral superfield is given by

$$d[\sqrt{Z}\Psi] = 1 + \frac{\gamma_c}{2} = \frac{3}{2}(1 - \frac{N}{N_f}) \tag{131}$$

the mass dimension of the scalar chiral superfield is not less than one, when $N_f \geq 3N$. The meson field is condensed in the non-perturbative low energy region, and its mass dimension is

$$d[M_g^f] = 3(1 - \frac{N}{N_f}) = \frac{3}{2}R[M_g^f] \tag{132}$$

which is a special example of a general theorem: At conformal fixed point, the mass dimension of scalar field is not less than 1 and the mass dimension of gauge invariant operator $d[\mathcal{O}]$ is related to the R charge of itself$R[\mathcal{O}]$ by

$$d[\mathcal{O}] \geq \frac{3}{2}R[\mathcal{O}] \tag{133}$$

which is saturated for chiral operators.

(I)$\frac{3}{2}N < N_f < 3N$

It is straight to get the ranges of the mass dimension of meson field,

$$1 < d[M_g^f] < 2 \quad \text{for} \quad \frac{3}{2}N < N_f < 3N \tag{134}$$

thus, the meson behave like scalar for $N_f \sim \frac{3}{2}N$, since $d[M_g^f] \sim 1$. To be brief, with $\frac{3}{2}N < N_f < 3N$ the physical gauge coupling is RG flowing to a non-trivial fixed point in the IR, thus the N=1 SUSY gauge theory with N_f flavors in the range is a IR free electric gauge theory.

(II)$N_f = 0$

In the following, lets consider a *pure* $SU(N)$ super Yang-Mills with all chiral superfield absent($N_f = 0$).

$$\mathcal{L} = \frac{1}{8\pi}\text{Im}(\tau \text{Tr} \int d^2\theta \mathcal{W}^\alpha W_\alpha) \tag{135}$$

Written in component fields, we have

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{g^2}{8\pi^2}\frac{i\theta}{4}F^a_{\mu\nu}\tilde{F}^{\mu\nu a} - \frac{i}{g}\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + \frac{1}{2g^2}D^aD^a \tag{136}$$

In this situation, we can not combine $U(1)_R^{'}$ and $U(1)_A$ together to construct anomaly free $U(1)_R$. The only left $U(1)_R^{'}$ symmetry is nothing but Z_{2N}, just a discrete subgroup of $U(1)_R$, which keep the pure gauge theory gauge invariant(The vacuum phase θ rotate integer times of 2π).

If gaugino condensation happens dynamically in the sense that the bilinear $\text{Tr}(\lambda^\alpha\lambda_\alpha)$ in the superpotential obtain a non-zero vacuum expectation value, namely $< \text{Tr}(\lambda^\alpha\lambda_\alpha) > \neq 0$, which can be reached by integrating out the glueball superfield $S = -\frac{1}{32\pi^2}\text{Tr}[W^\alpha W_\alpha]$ in the Veneziano-Yankielowicz superpotential

$$W_{eff} = NS - NS\log\frac{S}{\Lambda^3} \tag{137}$$

This results in the non-perturbative dynamical super-potential

$$W \sim < \text{Tr}\lambda^\alpha\lambda_\alpha > = N\Lambda^3 e^{i\frac{2\pi k}{N}}, \text{where} \quad k = 1, \ldots, N \tag{138}$$

where Λ is the dynamical energy scales of the pure gauge theory with N phases of the vacuum appear(In analog to the goldstone degree of freedom due to spontaneously symmetry breaking of continuous symmetry)since the quantum discrete Z_{2N} R symmetry is broken down to Z_2 symmetry due to gaugino condensation.

(III)$1 \leq N_f \leq N - 1$

When flavors are added, $1 \leq N_f \leq N-1$, the gauge invariant scalar(gauge singlet) chiral super-field $M_g^f \equiv \Phi^f \tilde{Phi}_g$ made out of chiral superfield transforms as a bi-fundamental representation $(\Box, \Box)$ of $SU(N)_f \times SU(N_f)$. The non-perturbative dynamical superpotential are constructed with non-perturbative energy scale Λ, and holographic gauge invariant functions(operators made out of the matter super-fields) of $\det M$ and holomorphic gauge coupling τ, which is just the Affleck-Dine-Seiberg(ADS) superpotential

$$W = (N - N_f)(\frac{\Lambda^{b_0}}{\det M})^{\frac{1}{N-N_f}} \tag{139}$$

for SUSY QCD. When gauge singlet M_g^f(with N_f^2 components) obtain vacuum expectation value and $\det M \neq 0$, the gauge symmetry $SU(N)$ is broken down to $SU(N - N_f)$, until $N_f = N - 1$, when the gauge symmetry is totally broken.

TABLE II: $N_f = N$ Electric theory for $M = \Lambda^2$, $B_f = \tilde{B}^g = 0$, chiral symmetry breaking to custodial symmetry

	$SU(N)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Φ^f	$\square$	$\square$	1	0
$\tilde{\Phi}_f$	$\square$	$\square$	-1	0
M^f_g	1	$N_f^2 - 1$	0	0
B_f	1	1	N	0
$\tilde{B}^g$	1	1	$-N$	0

TABLE III: $N_f = N$ Electric theory for $M = 0$, $B_f = -\tilde{B}^g = \Lambda^N$, $U(1)_B$ is broken

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)_R$
Φ^f	$\square$	$\square$	1	0
$\tilde{\Phi}_f$	$\square$	1	$\square$	0
M^f_g	1	$\square$	$\square$	0
B_f	1	1	1	0
$\tilde{B}^g$	1	1	1	0

(IV) $N_f = N$

The interesting thing happens when $N_f = N$, not only that the gauge singlet M^g_f becomes mesons, but also two additional extra gauge invariant gauge singlet, namely the baryons and anti-baryons.

$$B = \det \Phi \quad \tilde{B} = \det \tilde{\Phi} \tag{140}$$

The total degree of freedom of gauge invariant complex number become $N_f^2 + 2$ by taking account meson, baryon and anti-baryon together. On the other hand, the gauge symmetry $SU(N)$ is totally broken, so that the only $2NN_f - (N^2 - 1) = N^2 + 1$ massless(unbroken) degree of freedom left($2N_f$ from scalar chiral superfields, N from $color$). Thus there is one extra massless(unbroken) degree of freedom, which is a constraint relating the meson, baryon and anti baryon, turning out to be

$$\det M - B\tilde{B} = \Lambda^{2N} \tag{141}$$

, and called *quantum moduli space*. This configuration of quantum vacuum state is deformed different from the classical moduli space($\Lambda = 0$)

$$\det M - B\tilde{B} = 0 \tag{142}$$

. At the point, it is worthy noticing that the quantum moduli space does not have solutions with all moduli vanishing, thus it is smooth everywhere, therefore we need at least $\det T \neq 0$ or $B, \tilde{B} \neq 0$, which means there is no point on the moduli space which preserves all the global $SU(N)_f \times SU(N)_f \times U(1)_B \times U(1)_R$ symmetry. While the classic moduli space contains a singular point where all gauge invariant moduli $\det M$, B and $\tilde{B}$ vanish, therefore at the singularity, we have some zero expectation values of chiral superfields, since $\det M = 0$, associating with massless vector fields. To be brief, the classical singularity are usually associated some fields becoming massless. Furthermore, for SUSY QCD, $b_0 = 3N - N_f = 2N$ and $\Lambda^{2N} \sim e^{-\frac{8\pi^2}{g^2}}$, therefore the quantum deforming comes from a single instanton. In summary, the classical singularity get smoothed out by non-perturbative quantum effects in SUSY gauge theory.

We can make a compariment with SUSY QCD and QCD at the quantum level at the moments. For examples, when $\det T = \Lambda^{2N}$ and $B = \tilde{B} = 0$, at the quantum level, the gauge symmetry $SU(N)_f \times SU(N)_f \times U(1)_B \times U(1)_R$ of SUSY QCD is broken to $SU(N)_D \times U(1)_B \times U(1)_R$, where D stands for diagonal, in analog to the chiral symmetry breaking happens in QCD. The difference here is that we could exactly calculate the non-perturbative dynamics and the vacuum structure at SUSY QCD.

One the other hand, we know that the one loop running of β function for N=1 SUSY QCD with N_f flavors has the coefficient$\sim -b_0 = -(3N - N_f)$. Thus at one loop level, the gauge coupling is relevant(at low energy) for $N_f < 3N$, irrelevant(in other words, UV free) for $N_f > 3N$ and marginal(β function is vanishing and gauge coupling is no long running) for $N_f = 3N$. The above discussion is based at the level of perturbative theory, where the couplings at tree level is not renormalized. However, non-perturvative effects could modify both the gauge couplings and couplings of operators at tree level which could turn the corresponding operator into relevant one.

III. SEIBERG DUALITY

With above intuitive observation for N=1 $SU(N)$ super Yang-Mills with N_f flavor, Seiberg propose following conjectures[4]: (a)The N=1 $SU(N)$ SUSY electric gauge theory with N_f flavor(Φ and $\tilde{\Phi}$) with $\frac{3}{2}N < N_f < 3N$ flows to a non-trivial fixed point in the IR. (b)For above electric gauge theory, there is dual magnetic N=1 $SU(N-N_f)$ gauge theory with N_f flavors(ψ and $\tilde{\phi}$) and gauge invariant scalar chiral superfield $\tilde{M}_f^g$ with tree level supperpotential

$$W_{tree} = y\phi^f \tilde{M}_f^g \tilde{\phi}_g \tag{143}$$

The number of the flavors (N_f, i.e., in analog to the number quarks in QCD) are physical and furthermore are the same for the electric theory and its magnetic electric theory, thus they both have the same *global gauge symmetry*, namely $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$. While the gauge symmetry could be different, since gauge symmetries are symmetries which relate different redundant degree of freedom and thus not a dynamical symmetry. The physical variables involved in physical observables are gauge invariant. The dual theory means the electric and magnetic theory are two sides of one coin, they are going to be description of the same theory.

For the dual magnetic gauge theory, define $\bar{N} = N_f - N$, we have

$$3\bar{N} > N_f > \frac{3}{2}\bar{N} \quad \text{for} \quad \frac{3}{2}N < N_f < 3N \tag{144}$$

Thus according to the first conjecture of Seiberg, the dual pure magnetic theory would also flow to a conformal fixed point. For a given arbitrary number of flavors, the electric and magnetic gauge theories have different ranks, since in general $N \neq \bar{N}$, except for $N_f = 2N$, thus basicly speaking, the electric and magnetic gauge theories go flow to different conformal fixed point. In order to have the same conformal fixed point, flow to which for both electric and magnetic theory, the dual of the magnetic theory(which is a electric theory)must have a rather relevant tree level superpotential to push the fixed point together.

The $R^{'}$ charges of the chiral super fields Φ, $\tilde{\Phi}$ are zero, thus the anomaly free R charge are

$$R[\Phi] = R[\tilde{\Phi}] = 1 - \frac{N}{N_f} \tag{145}$$

according to Eq.(111).

TABLE IV: Dual magnetic theory

	$SU(N_f-N)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
ϕ^f	$\square$	$\square$	1	$\frac{N}{N_f-N}$	$\frac{N}{N_f}$
$\tilde{\phi}_f$	$\square$	1	$\square$	$-\frac{N}{N_f-N}$	$\frac{N}{N_f}$
$\phi^f\tilde{\phi}_g$	1	$\square$	$\square$	0	$2\frac{N}{N_f}$

Here in the dual theory with a gauge symmetry $SU(\bar{N})$, we have the anomaly free R charge

$$R[\phi] = R[\tilde{\phi}] = \frac{N}{N_f} \tag{146}$$

thus $R[\phi^f\tilde{\phi}_g] = 2\frac{N}{N_f}$, in addition $d[\phi^f\tilde{\phi}_g] = \frac{3}{2}R[\phi^f\tilde{\phi}_g] = 3\frac{N}{N_f}$, since $\phi^f\tilde{\phi}_g$ is a gauge invariant chiral operator and we have

$$d[\phi^f] = d[\tilde{\phi}_g] = \frac{3N}{2N_f} \tag{147}$$

. Note, by observing Eq.(145) and Eq.(147), it is clear that the baryon charge(being $\pm\frac{N}{N-N_f}$) of ψ^f and $\tilde{\phi}_g$ with $\bar{N} = N - N_f$ color in dual magnetic theory is normalized with respect to the charge(being ± 1) for Φ^f and $\tilde{\Phi}_g$ with N color.

Therefor the mass dimension of the operator in the tree leve superpotential of the dual magnetic theory in Eq.(143), is $d[\phi^f\tilde{M}_f^g\tilde{\phi}_g] = 1 + 3\frac{N}{N_f}$, and

$$3 > d[\phi^f\tilde{M}_f^g\tilde{\phi}_g] > 2 \quad \text{for} \quad \frac{3}{2}N < N_f < 3N \tag{148}$$

thus the tree level superpotential derives the magnetic theory to the same conformal fixed point as the electric gauge theory. The tree level couplings has a mass diemension

$$d[y] = 3 - d[\phi^f \tilde{M}_f^g \tilde{\phi}_g] = 2 - 3\frac{N}{N_f} \tag{149}$$

Thus we get the mass dimension of magnetic scalar $\tilde{M}_f^g$,

$$2 > d[\tilde{M}_f^g] > 1 \quad \text{for} \quad \frac{3}{2}N < N_f < 3N \tag{150}$$

Observering Eq.(134) and Eq.(150), the Seiberg duality maps the meson field M_g^f(the expectation values of $\Phi^f \tilde{\Psi}_g$) in the UV to the magnetic scalar $\tilde{M}_f^g$ in the IR, but $d[M_f^g] = 2$ in the UV while $d[\tilde{M}_f^g] = 1$. Therefore, the meson and magnetic scalar must be rescaled by a scale μ with mass dimension,

$$M_f^g = \mu \tilde{M}_f^g \tag{151}$$

The tree level superpotential in the magnetic theory becomes

$$W_{tree} = \frac{y}{\mu}\phi^f M_f^g \tilde{\phi}_g \tag{152}$$

Next, let's think about how to go back to the electric theory. Consider the dual of the magnetic theory, the dual of the magnetic quark $(\phi\tilde{\phi})$ form a dual "meson", $\tilde{\tilde{M}}_f^g$ with dimension 2, then the tree level superpotential in the dual to the magnetic theory is

$$W_{tree} = \frac{y}{\tilde{\mu}}\Phi^f \tilde{\tilde{M}}_f^g \tilde{\Phi}_g + \frac{y}{\mu} M_f^g \tilde{\tilde{M}}_f^g \tag{153}$$

Once the meson get expectation value and give mass to dual "meson" $\tilde{\tilde{M}}_f^g$, the dual "meson" can be integrate out and give

$$M_f^g = \frac{\tilde{\mu}}{-\mu}\Phi^f \tilde{\Phi}_g \tag{154}$$

thus the dual "meson" and the meson are the same thing if and only if $\tilde{\mu} = -\mu$.

Now consider the non-perturbative energy scale denoted by Λ_e of the electric theory that of the dual magnetic theory by Λ_m. The electric and magnetic variables are related by a scale with a unite mass dimension, the full relation between electric and magnetic theory must be

$$\Lambda_e^{3N-N_f}\Lambda_m^{3\bar{N}-N_f} = (-1)^{\bar{N}}\mu^{N_f} \tag{155}$$

so that for the relation between magnetic and dual magnetic theory is

$$\Lambda_m^{3\bar{N}-N_f}\Lambda_e^{3N-N_f} = (-1)^{N}\tilde{\mu}^{N_f} \tag{156}$$

and both above identities are the same relations considering that $\tilde{\mu} = -\mu$.

For the electric and magnetic theory gauge couplings $g_e(\mu)$ and $g_m(\mu)$ at the cutoff μ, we have

$$\Lambda_e^{3N-N_f} = \mu^{3N-N_f} e^{-\frac{8\pi^2}{g_e(\mu)^2}} \quad \text{and} \quad \Lambda_m^{3\bar{N}-N_f} = \mu^{3\bar{N}-N_f} e^{-\frac{8\pi^2}{g_m(\mu)^2}} \tag{157}$$

If the electric gauge theory is strongly coupled at some high energy μ, so that $\Lambda_e < \mu$, then from the relation between electric and magnetic theory, we find $(-1)^{N_f-N}(\frac{\mu}{\Lambda_m})^{2N_f-3N} < 1$, or

$$|\frac{\mu}{\Lambda_m}| < 1 \tag{158}$$

the magnetic gauge theory become weakly coupled at that scale(and weakly coupled at low energy). The Seiberg duality means what was mysterious in the electric theory is simple in the magnetic theory, and vice versa.

In the last section, we have discuss the physics of SUSY QCD when the flavor number are in the ranges $0 \geq N_f \leq N$ as well as $\frac{3}{2}N < N_f < 3N$. In the following, we would discuss all of other possibility.

(V)$N_f = N + 1$

TABLE V: $N_f = N+1$ Electric theory

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Φ^f	$\square$	$\square$	1	1	$\frac{1}{N_f}$
$\tilde{\Phi}_f$	$\square$	1	$\square$	-1	$\frac{1}{N_f}$
M_g^f	1	$\square$	$\square$	0	$\frac{2}{N_f}$
B_f	1	$\square$	1	N	$\frac{N}{N_f}$
$\tilde{B}^g$	1	1	$\square$	$-N$	$\frac{N}{N_f}$
$B_f\tilde{B}^g$	1	$\square$	$\square$	0	$\frac{2N}{N_f}$
$\det M$	1	1	1	0	2
Λ^{2N-1}	1	1	1	0	0

(a)massless case

If the flavor $N_f = N+1$, we still have meson $M_g^f = \Phi^f\tilde{\Phi}_g$ with $(N+1)^2$ degree of freedom, in addition $N_f = N+1$ baryons B_f and $N_f = N+1$ anti-baryons $\tilde{B}^g$, with the definition

$$B_f = \epsilon_{ff_1\ldots f_N}\Phi^{f_1}\ldots\Phi^{f_N} \quad \tilde{B}^g = \epsilon^{gg_1\cdots g_N}\tilde{\Phi}_{g_1}\ldots\tilde{\Phi}_{g_N} \tag{159}$$

with a total degree of freedom $N_f^2 + 2N_f = N^2 + 4N + 3$ for the theory. By imposing 't Hooft anomaly so that on one hand, the gauge symmetry $SU(N)$ is anomaly free with flavor degree of freedom from $SU(N_f)\times SU(N_f)\times U(1)_B\times U(1)_R$, and on the other hand the global symmetry $SU(N_f)\times SU(N_f)\times U(1)_B\times U(1)_R$ is UV and IR anomaly with color degree of freedom from $SU(N)$. We can construct the unique gauge and flavor invariant and holomorphic dynamical superpotential with R charge 2 and mass-dimension 3

$$W = \frac{1}{\Lambda^{2N-1}}(B^f M_f^g\tilde{B}^g - \det M) \tag{160}$$

Minimize the superpotential with gauge invariant variables M_f^g, B^f and $\tilde{B}_g$, we get constraints

$$\det M(M^{-1})_g^f - B^f\tilde{B}_g = 0, \quad M_g^f B_f = 0, \quad M_g^f\tilde{B}^g = 0 \tag{161}$$

with $N_f^2 + 2N_f = N^2 + 4N + 3$ degree of freedom, which are exactly the number of degree of freedom of the gauge invariant moduli in M_g^f, B_f and $\tilde{B}^g$. Thus the gauge symmetry is unbroken and the (electric) theory is confined with massless mesons and massless baryons, the confinement does not lead to chiral symmetry breaking. On the other hand, it is clear that the non-perturbative energy scale does not appear in the constraint so that these configuration of mouli are just classical one and they must have singularities which associated with massless gauge vectors, as analyzed before. In other words, the quantum moduli space of the massless theory with $N_f = N+1$ flavors is the same as the classical theory and the global symmetry is preserved at the singularity.

(b) massive cases

For theory with massive mesons, we can obtain the constraints by integrating in one more flavor to the $N_f = N$ theory as shown in Eq.(141). The result is

$$\det M(M^{-1})_g^f - B^f\tilde{B}_g = m_g^f\Lambda^{2N-1} \tag{162}$$

. Thus for the massive meson case, the quantum moduli space is different from the classical moduli space, since the non-zero meson mass $m_g^f \neq 0$. (Note: In the massless situation, the quantum moduli space and classical moduli space is still different for $N_f = N$ cases, namely Eq.(141) is still kept for massless cases. This is the difference between $N_f = N$ and $N_f = N+1$.)

For theory with massive baryons, due to non-zero M_f^g give the mass to the baryons thus the baryons can be integrated out and the only gauge invariant operators left are mesons M_g^f with $N_f^2 = (N+1)^2$ degree of freedom, namely

$$\det M(M^{-1})_g^f = 0 \tag{163}$$

with baryons are removed from the original constraints. The low energy theory with only T_g^f has no superpotential, if $M_g^f \gg \Lambda^2$, the descriptions based on the superpotentail160 is in-valid.

(VI)$N+2 \leq N_f \leq \frac{3}{2}N$ The electric gauge theory with flavors in this range is complicated. However the Seiberg duality does work here, if we define $\bar{N} = N_f - N$, we have

$$N_f \geq 3\bar{N} \quad \text{for} \quad N+2 \leq N_f \leq \frac{3}{2}N \tag{164}$$

(VII) $\frac{3N}{2} < N < 3N$ the magnetic theory is $SU(\bar{N})$ and it is IR free when $N_f \geq \bar{N}$, thus for the electric gauge theory with flavors in the range $N+2 \leq N_f \leq \frac{3}{2}N$ is a weakly coupled magnetic theory, and the electric theory in this range is (IR) irrelevant. This can be observed that in the range, the operator $\phi^f \tilde{M}_f^g \tilde{\phi}^g$ in the superpotential has the mass dimension $d \geq 3$, so that the coupling y in front of the operator has a mass dimension $d[y] \leq 0$. Therefore the gauge couplings of the dual magnetic theory simply is RG flowing to zero in IR.

(VII)$N_f \geq 3N$

For N=1 SUSY of SU(N) gauge theory with flavor N_f in this range, the critical anomalous dimension in Eq.(130) is in the range

$$0 \leq \gamma_c \leq 1 \quad \text{for} \quad 3N \leq N_f < \infty \tag{165}$$

so that the mass dimension of the chiral field is

$$1 \leq d[\Phi] \leq \frac{3}{2} \quad \text{for} \quad 3N \leq N_f < \infty \tag{166}$$

and that of the meson field is

$$2 \leq d[M_g^f] \leq 3 \quad \text{for} \quad 3N \leq N_f < \infty \tag{167}$$

From NSVZ β function in Eq.(128), we always have $\beta(g) > 0$, thus the theory is IR free and the low energy theory contains free quarks and gluons.

IV. SEIBERG-WITTEN THEORY

Our motivation here is to study strongly coupled gauge theory from the weakly coupled physics via duality in a quantitative approach.

We begins from N=2 SUSY Lagrangian,

$$\mathcal{L} = \frac{1}{8\pi}[\tau \text{Tr}(2\int d^4\theta \Phi^\dagger e^{-2V}\Phi + \int d^2\theta \mathcal{W}^\alpha \mathcal{W}_\alpha)] \tag{168}$$

with scalar multiplet $\Phi = (\varphi, \psi_\alpha)$ and the vector multiplet $\mathcal{W}_\alpha = (\lambda_\alpha, A_\mu)$ as N=1 SUSY, in addition, both Φ and $\mathcal{W}_\alpha$ belong the same vector multiplet and also transform in the adjoint representation.

After some algebra, the Lagrangian is

$$\begin{aligned}\mathcal{L} = &\text{Tr}[\tfrac{1}{g^2}D_\mu\varphi^\dagger D^\mu\varphi - \tfrac{i}{g^2}\bar{\psi}\bar{\sigma}^\mu D_\mu\psi] + \tfrac{i}{g^2}\sqrt{2}\{\lambda,\psi\}\varphi^\dagger - \tfrac{i}{g^2}\sqrt{2}\{\bar{\lambda},\bar{\psi}\}\varphi^\dagger] \\ &+\text{Tr}[-\tfrac{1}{4g^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + \tfrac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \tfrac{i}{g^2}\lambda\sigma^\mu D_\mu\bar{\lambda}] - V\end{aligned} \tag{169}$$

where the potential including F terms and D terms is

$$V = -\text{Tr}[\frac{1}{g^2}D[\varphi^\dagger,\varphi] + \frac{1}{g^2}F^\dagger F + \frac{1}{2g^2}DD] \tag{170}$$

by integrating out the auxiliary fields, it can be simplified as

$$V = \frac{1}{2g^2}\text{Tr}[\varphi^\dagger,\varphi]^2 \tag{171}$$

From which we get the classical moduli space by minimize the potential and given by

$$\varphi = \varphi^a T^a \tag{172}$$

where the index a runs only over the commutative part of the group space, thus are the generators of the Cartan subalgebra. For N=2 SUSY $SU(N)$ gauge theory, its rank is $N-1$ and h runs from 1 to $N-1$. The $SU(N)$ gauge symmetry is broken by the moduli shown above down to $\Pi_{i=1}^{N-1}U(1)^i$, i.e., for N=2 $SU(2)$ SUSY, its rank is one and there is only one carton generator, namely $T^3 = \sigma^3$. Although the gauge symmetry is broken from $SU(2)$ down to $U(1)$, the N=2 SUSY symmetry remains unbroken.

The classical moduli space(manifold of the gauge inequivalent vacuum) can be parameterized as a complex parameter,

$$u = \frac{1}{2}\text{Tr}[\varphi^2] = \frac{1}{2}a^2 \tag{173}$$

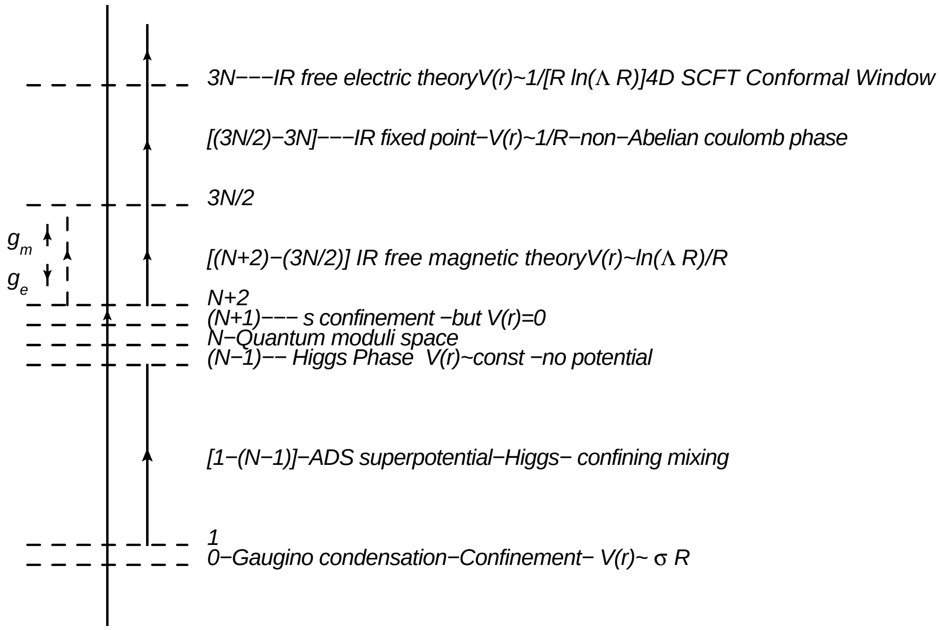

FIG. 1: Phase diagrams for N=1 SU(N) SUSY QCD classified by Seiberg duality.
(I-1)$\frac{3N}{2} < N < 3N$, non-Abelian Coulomb phase, the gauge coupling does not run, with a Coulomb potential $V(r) \sim \frac{1}{R}$, an interacting conformal theory, no particle description, only γ is physical, conformal window.
(I-2)$N \sim 3N$, a perturbative infrared fixed point.
(II)$N_f = 0$, gaugino condensation, Confining phase(means its infrared physics can be described exactly in terms of gauge invariant composites and its interactions), linear potential $V(r) \sim \sigma R$, σ is the string tension(constant mass per unite length.)
(III-1)$1 \leq N_f \leq N-2$, gaugino condensation generates ADS superpotential
(III-2)$N_f = N-1$,instantons can generates ADS superpotential, Higgs phase, the gauge bosons are massive so that there is no long range forces. (IV)$N_f = N$, quantum moduli space is deformed to be continous, confining in the sense that all of the massless degrees of freedom(meson and baron) are color singlets particles.
(V)$N_f = N+1$, confining phase without chiral symmetry breaking and with a non-vanishing confining superpotential("s confinement"), since quantum moduli space is singular as the classic moduli space, all particles(including meson and baryon) are color singlet. Gauge symmetry is unbroken, gluons and gluinos are massless, global symmetry is preserved,
(VI)$N+2 \leq N_f \leq \frac{3}{2}N$, free dual magnetic gauge theory, lose asymptotic freedom. Static magnetic monopole grows logarithmically due to the renormalization of massless electrons loops. $V(r) \sim \frac{\log R\Lambda}{R}$. The electric theory is weakly coupled near $N_f = 3N$ and becomes strong when N_f decrease, while the dual magnetic theory is weakly coupled near $N_f = \frac{3}{2}N$ and becomes stronger when N_f increase.
(VII)$N_f \geq 3N$, free electric gauge theory, the couplings grows in the IR, lose asymptotic freedom, a weakly coupled low-energy effective theory. Static electrons are renormalized by massless monopole loops. $V(r) \sim \frac{1}{R \log R\Lambda}$.

which are gauge invariant under the Weyl reflection $\varphi \to -\varphi$. Note that the variable u can take any complex numbers thus the classical moduli space is a complex plane, together with a point at infinity that compactifying the complex plane, we get a Riemann sphere S^2. The classical moduli space has singularity when $u = 0$, means $\varphi = 0$ and the gauge bosons are all massless.

The most general renormalizable Lagrangian can be written in term of N=2 chiral super-filed Ψ in superspace, with superspace coordinates as $(x, \theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2)$.

$$\mathcal{L} = \frac{1}{4\pi}(\tau \mathrm{Tr} \int d^2\theta_1 d^2\theta_2 \frac{1}{2}\Psi^2) \tag{174}$$

$$\Psi = \Phi(y_2, \theta_1) + \sqrt{2}\theta_2^\alpha \mathcal{W}_\alpha(y_2, \theta_1) + \tilde{\theta}_2^2(\Phi^\dagger(y_2 - i\theta_1\sigma\bar{\theta}_1, \theta_1, \bar{\theta}_1)e^{2gV(y_2 - i\theta_1\sigma\bar{\theta}_1, \theta_1, \bar{\theta}_1)})_{\bar{\theta}_1, \bar{\theta}_1}$$
$$y_2^\mu = x^\mu + i\theta_1\sigma^\mu\bar{\theta}_1 + i\theta_2\sigma^\mu\bar{\theta}_2 \tag{175}$$

satisfying $\bar{D}^1_{\dot{\alpha}}\Psi = 0$ and $\bar{D}^2_{\dot{\alpha}}\Psi = 0$, with respect to θ_1 and θ_2, respectively.

The holomorphic gauge coupling

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \tag{176}$$

The general Lagrangian in effective low energy(non-linear σ model) N=2 SUSY can be written with a general function of the chiral superfield of Ψ, called *prepotential* $\mathcal{F}(\Psi)$,

$$\mathcal{L} = \frac{1}{4\pi}(\tau \mathrm{Tr} \int d^2\theta_1 d^2\theta_2 \mathcal{F}(\Psi)) \tag{177}$$

(note: if we choose $\mathcal{F}(\varphi) = \frac{1}{2}\mathrm{Tr}[\varphi^2]$), we reproduce the standard super Yang-Mills in Eq.(169).
after expanding, it is

$$\mathcal{L} = \frac{1}{4\pi}(\int d^4\theta(\Phi^\dagger e^{2gV})^a \mathcal{F}_a(\Psi) + \frac{1}{2}\int d^2\theta \mathcal{F}_{ab}(\Psi)\mathcal{W}^{a\alpha}\mathcal{W}^b_\alpha) \tag{178}$$

where

$$h \equiv \mathcal{F}_a(\Phi) = \frac{\partial F}{\partial \Phi^a}, \quad \tau \equiv \frac{\partial h}{\partial \Phi^b} = \mathcal{F}_{ab}(\Phi) = \frac{\partial^2 F}{\partial \Phi^a \partial \Phi^b} \tag{179}$$

which gives the Kahler potential(real-valued function of both Φ and its conjugate $\Phi^\dagger$)

$$K = \mathrm{Im}(\Phi^{\dagger \mathrm{a}} \mathcal{F}(\Phi)) \tag{180}$$

and

$$g_{ab} = \mathrm{Im}(\frac{\partial^2 \mathrm{F}}{\partial \Phi^{\mathrm{a}} \partial \Phi^{\mathrm{b}}}) \tag{181}$$

from the Kahler metric

$$ds^2 = \frac{\partial^2 K}{\partial \Phi^{\dagger a}\partial \Phi^b} d\Phi^a d\Phi^b = \mathrm{Tr}[\tau] d\Phi d\Phi \tag{182}$$

The terms that we are interested is the second term in the Eq.(178), expressed as

$$\mathcal{L} = \frac{1}{16\pi}\mathrm{Im}(i\frac{\partial^2 \mathcal{F}(\varphi)}{\partial \varphi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}) = \frac{1}{16\pi}\mathrm{Tr}[\tau(F_{\mu\nu}F^{\mu\nu} + i\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})] = \frac{1}{32\pi}\mathrm{Tr}[\tau(F_{\mu\nu} + i\tilde{F}_{\mu\nu})^2] \tag{183}$$

Thus for N=2 SUSY, the SUSY algebra is invariant under $U(2)_R = SU(2)_R \times U(1)_R$ symmetry, which $U(1)_R$ will contribute a anomaly like the chiral anomaly from $U(1)_A$. The Lagrangian is constrained and expressed in terms of a holomorphic function-prepotential $\mathcal{F}(\Phi)$. Both the gauge coupling and the Kahler potential are determined by the prepotential. Because the holomorphic gauge couplings is renormalized only at one loop and the one loop β function is exact, thus the exact prepotential in the perturbative region can be determined by using only one loop calculation.

Take N=2 SU(2) SUSY as an example, it turns out that the prepotential contains both perturbative and non-perturbative contributions expressing as follows

$$\mathcal{F} = \frac{i}{2\pi}\varphi^2 \log\frac{\varphi^2}{\Lambda^2} + \sum_{i=1}^{\infty} c_k(\frac{\Lambda}{\varphi})^{4i}\varphi^2 \tag{184}$$

where c_i are constant coefficients.

The perturbative part is the first part of the above equation, which is determined by requiring the shift of prepotential contributes exactly the chiral anomaly from the $U(1)_R$ symmetry.

$$\delta L = -\frac{\alpha}{4\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{185}$$

According to non-renormalization theorems for N=2 SUSY, one loop β functions is exact so that there is no higher(two loops or more) loop corrections to the $\mathcal{F}$ in the full perturbative result, the exact perturbative prepotential is of one loop logarithmic form

$$\mathcal{F}_{pert} = \frac{i}{2\pi}\varphi^2 \log\frac{\varphi^2}{\Lambda^2} \tag{186}$$

where Λ is the non-perturbative dynamical scale of $SU(2)$. This perturbative prepotential would correctly describe the dynamics of the theory in the local ultraviolet large φ region, however it can not give a good global description.

The second part of the equation comes from multi-instanton, which is non-perturbative, since amplitude of one single instanton is proportional to $e^{-\frac{8\pi^2}{g^2}}$ and it is small when couplings is weak and becomes more important when gauge coupling becomes stronger. It is well known that the dynamical scale is defined as

$$\Lambda^{b_0} = \varphi^{b_0} e^{-\frac{8\pi^2}{g^2}} = \varphi^{b_0} e^{2\pi\tau - i\theta} \tag{187}$$

where $b_0 = 2N - N_f$ for pure N=2 SUSY gauge theory with N_f flavors, thus for the pure $SU(2)$ SUSY, a k-instanton contribution is proportional to

$$e^{-\frac{8\pi}{g^2}k} = (\frac{\Lambda^4}{\varphi^4})^k \tag{188}$$

and the non-perturbative contribution to prepotential by summing over all k-instantons is

$$\mathcal{F}_{non-pert} = \sum_{i=1}^{\infty} c_i (\frac{\Lambda^4}{\varphi^4})^k \varphi^2 \tag{189}$$

which also restores the $U(1)_R$ symmetry. This non-perturbative prepotential would give important effect when the theory goes to infrared region, where the theory is strongly coupled and perturbative approach fails.

In the N=2 SUSY, a dyon(particles with both electric and magnetic charge $Z = n_m\varphi_D + n_e\varphi$) with mass M, are states bounded by the so called *BPS bound*, coming from the positivity of norms of helicity states of N=2 SUSY algebra.

$$M \geq \sqrt{2}|Z|, \quad \text{with} \quad Z = n_m\varphi_D + n_e\varphi \tag{190}$$

with a non-zero central charge Z for massive states, thus it is clear BPS states those are stable for which (n_m, n_e) are relatively prime.

On the complex plane, there is a $SL(2,Z)$ symmetry defining a complex structure, a periodic grid of 2-Dimensional lattice with periods ω_1 and ω_2, which implies that given a complex number z on the complex plane, we have $z \sim z + n\omega_1 + m\omega_2$. This grid with periodicity also defines a torus by identifying opposite sides of a given parallelogram with 4 different corners and 4 sides. Given a parallelogram with periods (ω_1, ω_2), another periods $(\omega_1^{'}, \omega_2^{'})$ is transformed by the matrix $M \in SL(2,Z)$(satisfying $ad - bc = 1$), namely

$$\begin{pmatrix} \omega_1^{'} \\ \omega_2^{'} \end{pmatrix} = M \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{191}$$

By defining the ratio $\tau \equiv \frac{\omega_1}{\omega_2}$, the two periods $(\tau, 1)$ and $(\tau^{'}, 1)$ are transformed under $SL(2,Z)$(In fact it is $SL(2,Z)/Z_2 \equiv PSL(2,Z)$, since $\pm M$ are equivalent to each other) and they define the same complex structure so that $tau^{'}$ is related to τ by

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{192}$$

which generates the S and T dualities as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{193}$$

The S and T duality together generates the $SL(2,Z)$, thus we expect a torus in the quantum moduli space of N=2 SUSY.

We can define the holomorphic gauge coupling $\tau(\varphi) = \frac{\partial^2\mathcal{F}}{\partial\varphi^2}$ and $h \equiv \frac{\partial\mathcal{F}}{\partial\varphi}$, so that $\tau = \frac{\partial h}{\partial\varphi}$. Similarly, we can make the same definition for the dual scalar field φ_D, and also $h_D = \frac{\partial\mathcal{F}}{\partial\varphi_D}$, and then the dual holomorphic gauge coupling becomes $\tau_D = \frac{\partial h_D}{\partial\varphi_D}$. Thus S-duality transformation maps τ to $\tau_D = -\frac{1}{\tau}$, implies

$$\frac{\partial h_D}{\partial\varphi_D} = \tau_D(\varphi_D) = -\frac{1}{\tau(\varphi)} = -\frac{\partial\varphi}{\partial h} \tag{194}$$

or

$$\varphi_D = h \quad h_D = -\varphi \tag{195}$$

The (h, φ) or equivalent $(\varphi_D, -h_D)$ are just periods on the complex plane, under the general $SL(2, Z)$ duality transformation. Under which the strongly interacting electrically charged particles can described by the weakly coupled magnetic monopoles.

We have already determine the prepotential $\mathcal{F}$ in the whole complex plane, thus in principle, we can determine the holomorphic gauge couplings $\tau = \frac{\partial^2 \mathcal{F}}{\partial \varphi^2}$ on the whole complex plane too, but we have to understand the singularities and monodromies and how the holomorphic gauge couplings changes around the singularities, since the singularities in moduli space correspond to shrinking cycles, and are associated with massless particles.

In the ultraviolet region($\varphi \to \infty$), the theory is asymptotic free and the microscopic $SU(2)$ is weakly coupled , the holomorphic gauge coupling(i.e. $\bar{\partial}\tau = 0$) is determined by definition as

$$\tau(\varphi) = \frac{\partial^2 \mathcal{F}_{pert}}{\partial \varphi^2} = \frac{i}{\pi}(\log \frac{\varphi^2}{\Lambda^2} + 3) \quad \text{for} \quad \varphi \to \infty \tag{196}$$

and

$$\varphi_D(\varphi) = h = \frac{\partial \mathcal{F}_{pert}}{\partial \varphi} = \frac{i\varphi}{\pi}(\log \frac{\varphi^2}{\Lambda^2} + 1) \quad \text{for} \quad \varphi \to \infty \tag{197}$$

Because $\mathcal{F}$ is holomorphic,

$$\text{Im}[\tau(\varphi)] = \text{Im}[\frac{\partial^2 \mathcal{F}(\varphi)}{\partial \varphi^2}] \tag{198}$$

is a *harmonic* function(we have $\partial_z \bar{\partial}_z \text{Tr}[\tau] = 0$, $z = x + iy$) and thus has no global minimum, hence can not globally stay positive(Im$\tau = \frac{4\pi}{g^2} > 0$) everywhere unless it is a constant as in the classic case. Thus this harmonic function must vanish somewhere, otherwise it is a constant. If the φ contour goes around the singularities, φ_D, φ do not return to their initial values thus are multi-valued functions, one has a non-trivial monodromy for them.

In the Infrared region($a \to \infty$), the theory is strongly coupled, and the non-perturbative prepotential effects.

$$\mathcal{F}_{non-pert} = \sum_{i=1}^{\infty} c_i (\frac{\Lambda^4}{\varphi^4})^k \varphi^2 \tag{199}$$

where the infinite summ represents instanton contributions and k is the instanton number and Λ is the dynamical scale of the theory.

The monodromy transformation

$$\begin{pmatrix} \varphi_D \\ \varphi \end{pmatrix} = M \begin{pmatrix} \varphi_D \\ \varphi \end{pmatrix} \tag{200}$$

is equivalent to transformation to change the quantum number of magnetic and electric particles.

$$(n_m, n_e) \to (n_m, n_e)M \tag{201}$$

It then turns out that the three singularities, namely ∞,φ_0(where $\varphi_D = h$=0) and $-\varphi_0$($\varphi_0 \neq 0$), with the monodromy matrix separately

$$M_{\varphi_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad M_{-\varphi_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \quad M_\infty = M_{\varphi_0} M_{-\varphi_0} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{202}$$

The singularity at ∞ is a electric charge particle with unite electric charge $(0, 1)$(as the left eigenvector of M_{φ_∞}), that at φ_0 is a magnetic monopole with unit magnetic charge $(1, 0)$ and that at $-\varphi_0$ is a dyon with both electric and magnetic charges with inverse sign$(1, -1)$.

In the complex plane, monodromies typically arise from differential equations with *meromorphic*(almost holomorphic, i.e., $\frac{1}{z}$) potential $V(z)$(If in the real space, it will be *periodic* potential $V(x + T) = V(x)$, where T is the period), having poles at $z_1, \ldots z_p$ and also ∞. When z goes once around any one of the poles, the differential equation does not change due to single-valuedness of $V(z)$. The solutions that along any continuous path around the z_i, must be the linear combination of two independent solutions $\psi_1(z)$ and $\psi_2(z)$.

$$\begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix} (z + e^{2\pi i}(z - z_i)) = M_i \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix} \tag{203}$$

with a 2 by 2 nontrivial monodromy matrix M_i for each of the poles of V that are at most 2nd order, namely the regular singular points. Considering the fact that there are non-trivial monodromies only at $\pm\varphi_0$ and ∞ implies that

φ_D, φ must satisfy the a 2nd order differential equation. It is straightforward to construct a potential with at most 2nd poles at the points ± 1(we have chosen normalization $\varphi_0 = 1$, so that $\varphi \pm 1$ should be replaced with $\frac{\varphi}{\varphi_0} \pm \varphi_0$) and ∞,

$$V(z) = -[\frac{1-\lambda_1^2}{(1+z)^2} + \frac{1-\lambda_2^2}{(1-z)^2} + \frac{1-\lambda_1^2-\lambda_2^2+\lambda_3^2}{(1-z)(1+z)}] \tag{204}$$

with double poles at ± 1 and ∞, and corresponding residues are $(1-\lambda_i^2)$ for $i = 1, 2, 3$.

The solutions(assuming $\lambda_i \geq 0$) are

$$\psi(z) = (1+z)^{\frac{1}{2}(1-\lambda_1)}(z-1)^{\frac{1}{2}(1-\lambda_2)} g((z+1)/2) \tag{205}$$

where there are tow independent solutions

$$\begin{aligned} g_1(z) &= (-z)^{-\frac{1}{2}(1-\lambda_1-\lambda_2+\lambda_3)} F(\tfrac{1}{2}(1-\lambda_1-\lambda_2+\lambda_3), \tfrac{1}{2}(1+\lambda_1-\lambda_2+\lambda_3), 1+\lambda_3; z^{-1}) \\ g_2(z) &= (1-z)^{\lambda_2} F(\tfrac{1}{2}(1-\lambda_1+\lambda_2-\lambda_3), \tfrac{1}{2}(1-\lambda_1+\lambda_2+\lambda_3), 1+\lambda_2; 1-z) \end{aligned} \tag{206}$$

where $F(a, b, c; z)$ is the hypergeometric function, one of the solutions of the hypergeometric differential equation,

$$z(1-z)\frac{\partial^2 f(z)}{\partial z^2} + [c-(1+a+b)z]\frac{\partial f(z)}{\partial z} - abf(z) = 0 \tag{207}$$

. The two independent solutions, namely $g_1(z)$,$g_2(z)$ have simple monodromy properties around $z = \infty$ and $z = 1$ separately, thus they can be identified with φ_D and φ. (a)$z \to \infty$, $V(z) \sim \frac{1-\lambda_3^2}{z^2}$, the $g_{1,2}(z)$ behave asymptotically as $z^{(1\pm\lambda_3)/2}$ for $\lambda_3 \neq 0$, and as $\sqrt{z}$ and $\sqrt{z}\log z$ for $\lambda_3 = 0$. (b)$z \to 1$, $\lambda_3 = 0$, $V(z) \sim 1/(1-z)^2$. The 2nd order differential is the so called Picard-Fuchs equation. It turns out that once we choose the parameters $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$, and then the potential $V(z) \sim \frac{1}{1-z^2}$, the two independent solutions are

$$\begin{aligned} \varphi_D = i\psi_2(z) = i\tfrac{u-1}{2}F(\tfrac{1}{2}, \tfrac{1}{2}, 2; \tfrac{1-u}{2}) &= \tfrac{\sqrt{2}}{\pi}\int_1^u dx \tfrac{\sqrt{x-u}}{\sqrt{x^2-1}} = \tfrac{\sqrt{2}}{\pi}\int_1^u \lambda = \tfrac{\sqrt{2}}{2\pi}\oint_{\gamma_1}\lambda \\ \varphi = -2i\psi_1(z) = \sqrt{2}(1+u)^{1/2}F(-\tfrac{1}{2}, \tfrac{1}{2}, 1; \tfrac{2}{1+u}) &= \tfrac{\sqrt{2}}{\pi}\int_{-1}^{-1} dx \tfrac{\sqrt{x-u}}{\sqrt{x^2-1}} = \tfrac{\sqrt{2}}{\pi}\int_{-1}^{+1}\lambda = \tfrac{\sqrt{2}}{2\pi}\oint_{\gamma_2}\lambda \end{aligned} \tag{208}$$

where γ_1 is a circle goes from $(1, u)$ on the first sheet and back from $(u, 1)$ on the second sheet, γ_2 is a circle goes once around the cut $(1, u)$ and $(-1, 1)$, and the so-called Seiberg-Witten differential $\lambda = \frac{\sqrt{x-u}}{\sqrt{x^2-1}}dx$, and these integrals over the periods, $\gamma_{1,2}$ of the torus-$\varphi_D = \oint_{\gamma_1}\lambda, \varphi == \oint_{\gamma_2}\lambda$, are called *period integrals*. There are two square-root branch cuts run from -1 to $+1$ and from u to ∞. The Riemann surface of the integrand is two-sheeted with the two sheets connected through the cuts.

The period integral along the elliptic curve on the genus-one Rieman surface

$$y^2 = (x^2-1)(x-u) \tag{209}$$

considered above has three singularities for $u = \pm 1$ and $u = \infty$, and have the correct monodromies in Eq.(202) with the vector (φ_D,φ) around the singularities. The monodromies are elements of the duality group $SL(2, Z)$, which acts on the vector φ_D,φ, and the eigen-value of the corresponding eigen-vector φ_D,φ, just label the electric and magnetic charge of massless particles(dynons) in the singularities(remember that the charges of the massless fields are the left eigenvectors of the respective monodromy matrices). As analyzed before, the singularity at ∞ is the perturbative singularity, since a unit electric charge particle become massless at this point of the moduli space, while the two singularities at $u = +1$ and $u = -1$ occur at the strong coupling, and are non-perturbative singularities arise because of a monopole or dyon becoming massless at the singularity $u = +1$ and $u = 1$ separately.

By defining the Seiberg-Witten meromorphic differential

$$\lambda \equiv \frac{ydx}{1-x^2} \tag{210}$$

then the vacuum expectation values of the scalar φ and of the dual scalar φ_D are determined as functions of the modulus u by integrating the meromorphic differential form λ over the appropriately chosen cycles γ_1, $\gamma_2 = \gamma_{1D}$ of the Riemann surface:

$$\begin{aligned} \varphi_D(u) &\equiv \tfrac{\partial\mathcal{F}(\varphi)}{\partial\varphi} = \tfrac{\sqrt{2}}{2\pi}\oint_{\gamma_1}\lambda = \tfrac{\sqrt{2}}{\pi}\int_{\Lambda^2}^u\lambda \\ \varphi(u) &= \tfrac{\sqrt{2}}{2\pi}\oint_{\gamma_2}\lambda = \tfrac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{+\Lambda^2}\lambda \end{aligned} \tag{211}$$

where

$$u \equiv < \text{Tr}\phi^2 >, \quad < \phi >= \varphi \frac{\sigma^3}{2} \tag{212}$$

and we have tracked back to the normalization conditions so that $u_0 = 1$ is recovered to be $u_0 = \Lambda^2$(We can also track further back, the Seiberg-Witten elliptic curve in Eq.(209) restores as $y^2 = (x^2 - u_0^2)(x-u) = (x^2 - \Lambda^4)(x-u)$, with three singularities $u = \pm u_0 = \pm\Lambda^2$ and $u = \infty$, the meromorphic differential become $\lambda = \frac{ydx}{\Lambda^4 - x^2}$).

Solving 2nd equation inversely to get $u(\varphi)$, insert into the 1st equation to obtain $\varphi_D(\varphi)$, integrate out with respect to φ, we find the prepotential $\mathcal{F}$ and thus the low-energy effective Lagrangian for N=2 SUSY theory is determined and so does the low energy dynamics. As u goes around ± 1 or ∞, the circles $\gamma_{1,2}$ are changed into a linear combination with integer coefficients with the $SL(2, Z)$ symmetry transforming on the complex plane.

Last but not the least, we get the holomorphic gauge couplings is the 2nd derivative of the the holomorphic prepotential $\mathcal{F}_{SW}$.

$$\tau(u) = \frac{d\varphi_D/du}{d\varphi/du} = \frac{\omega_D(u)}{\omega(u)} = \frac{\partial^2 \mathcal{F}_{SW}}{\partial\varphi^2} = \frac{4\pi i}{g^2(u)} + \frac{\vartheta(u)}{2\pi} \tag{213}$$

where ω_D and ω are the periods associated with the circle γ_1 and γ_2, separately. The equation describer the complex structure of the torus with guarantee $\text{Im}[\tau(u)] = \frac{4\pi}{g^2} > 0$ for the positivity of the effective gauge coupling (or for the positivity of metric on the moduli space). From the holomorphic gauge couplings, we can also establish the metric on the moduli space,

$$g_{z\bar{z}} \sim \frac{\partial\mathcal{F}}{\partial\varphi} \tag{214}$$

With above observation, the low-energy solution can be obtained by introducing an auxiliary Riemann surface of genus equal to the rank of the gauge group(i.e., for $SU(5)$, r=4, the genus is 4). All holomorphic quantities can be computed as elliptic integrals over the surface, which already encode the singularity structure of the theory, and turns out to be a hyper-elliptic surface in most cases,

$$y^2 = P(x, u_i, \Lambda) \tag{215}$$

where P is a polynomial in auxiliary parameter x, the coordinates on the moduli space u_i and the dynamical scale of the theory Λ. Thus to find the exact low-energy action is equivalent to find the the polynomial P that describing the hyper-elliptic surface, from which one can extract information about the dynamics of the theory, namely, the effective gauge couplings($\tau = \mathcal{F}^{''}$), the metric on the moduli space($\mathcal{F}^{'}$)and the spectrum of PBS states($M \sim Z = (n_m, n_e)\begin{pmatrix} \varphi_D \\ \varphi \end{pmatrix}$). Note, the hyper-elliptic surface obtained in the limit $\Lambda \to 0$ is singular *everywhere* on the moduli space and just reflect the fact that turning off the gauge couplings will result in additional massless gauge bosons independently of the vacuum expectation values of the scalars, since there is no Higgs mechanism in this limit. Therefore the dynamical scale Λ has to appear as a parameter of the full Seiberg-Witten theory so that the classical singularities must be smoothed out by physic effects proportional to non-zero Λ, except the singularity at 0, which indicating the existence of additional massless particles.

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APPENDIX A: DIMENSION OF IRREDUCIBLE REPRESENTATION, DENKIN INDEX AND ANOMALY COEFFICIENT OF CLASSICAL LIE GROUPS

Notation: $Irrep$ represents Irreducible representation **R**, $Dim(\mathbf{R})$ is dimension of Irreducible representation, $c(\mathbf{R}) = 2T(\mathbf{R})$ is Denkin index and $A(\mathbf{R})$ is anomaly coefficient.

TABLE VI: $SU(N)$

Irrep **R**	Dim(**R**)	c(**R**)=2T(**R**)	A(**R**)
[1]	N	1	1
Adj	N^2-1	$2N$	0
$[1,0]$	$\frac{N(N-1)}{2}$	$N-2$	$N-4$
[2]	$\frac{N(N+1)}{2}$	$N+2$	$N+4$
$[1,0,0]$	$\frac{N(N-1)(N-2)}{6}$	$\frac{(N-2)(N-3)}{2}$	$\frac{(N-3)(N-6)}{2}$
[3]	$\frac{N(N+1)(N+2)}{6}$	$\frac{(N+2)(N+3)}{2}$	$\frac{(N+3)(N+6)}{2}$
$[2,1]$	$\frac{N(N^2-1)}{3}$	N^2-3	N^2-9
$[2,0]$	$\frac{N^2(N^2-1)}{12}$	$\frac{N(N^2-4)}{3}$	$\frac{N(N^2-16)}{3}$
[4]	$\frac{N(N+1)(N+2)(N+3)}{24}$	$\frac{(N+2)(N+3)(N+4)}{6}$	$\frac{(N+3)(N+4)(N+8)}{6}$
$[2,1,0]$	$\frac{N(N+1)(N-1)(N-2)}{8}$	$\frac{(N-2)(N^2-N-4)}{2}$	$\frac{(N-4)(N^2-N-8)}{2}$

TABLE VII: Direct tensor product of two representation

Irrep **R**	Dim(**R**)=d(**R**)	c(**R**)=2T(**R**)	A(**R**)
$\mathbf{R}_1 \times \mathbf{R}_2$	$d(\mathbf{R}_1)d(\mathbf{R}_2)$	$d(\mathbf{R}_1)c(\mathbf{R}_2)+c(\mathbf{R}_1)d(\mathbf{R}_2)$	$d(\mathbf{R}_1)A(\mathbf{R}_2)+A(\mathbf{R}_1)d(\mathbf{R}_2)$

---

[1] Note: For N=4 SUSY, the beta function vanish so the gauge coupling does not run and stay marginal in the quantum theory as in the classical theory. The $SU(4)_R$ symmetry of N=4 SUSY is isomorphic to $SO(6)$, the sphere space with positive curvature. In terms of AdS/CFT correspondence, it is this $SO(6)$ isometry corresponds to N=4 R symmetry. On the other hand, The isometry of AdS_5, a space with constant negative curvature, is $SO(4,2)$, which is precisely the same group as the 4D conformal symmetry group(CFT)

[2] For N=8 SUSY, graviton and gravitino appear. For more detail, refer to Stefan's notes on supergravity.

[3] Note: For more detail on Instanton, see Yuhsin's note.

[4] Note: For the application of Seiberg Duality in dynamical SUSY breaking in meta-stable vacua, see Flip's note on *ISS model*(Intriligator, Seiberg, and Shih)in *BSM Journal Club, Fall 2009* for more detail. For more basic introduction to notations and terms on dynamical SUSY breaking, see Johannes's note on *Dynamical SUSY breaking*.

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