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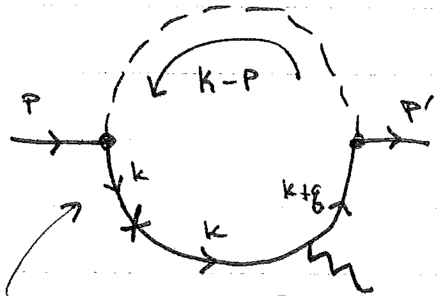
4D CALCULATION

2 MAY 2012



PREFACTOR : $i^4 \cdot \frac{v}{\sqrt{2}} Y^3$

$$\frac{\not k + \not q + m_{(m)}}{(k+q)^2 - m_{(m)}^2}$$

$$\frac{\not k}{k^2 - m_{(n)}^2} \qquad \frac{\not k + m_{(m)}}{k^2 - m_{(m)}^2} \qquad \times\left(\frac{1}{(k-p)^2 - m_h^2}\right)$$

$$\mathcal{M}^\mu = [\text{PRE}] \int d^4k \; \bar u(p') \left[\frac{\not k + \not q + m_{(m)}}{(k+q)^2 - m_{(m)}^2} \gamma^\mu \frac{\not k + m_{(m)}}{k^2 - m_{(m)}^2} \cdot \frac{\not k}{k^2 - m_{(n)}^2} \right] u(p)$$

$$\frac{1}{(k+q)^2 - m_{(m)}^2} \cdot \frac{1}{(k-p)^2 - m_h^2} \cdot \frac{1}{k^2 - m_{(m)}^2} \cdot \frac{1}{k^2 - m_{(n)}^2}$$

$$= \underbrace{\left(\frac{1}{k^2 - m_{(m)}^2}\right)^2 \frac{1}{k^2 - m_{(n)}^2} \frac{1}{k^2 - m_h^2}} \left(1 - \frac{2k\cdot q}{k^2 - m_{(m)}^2}\right)\left(1 + \frac{2k\cdot p}{k^2 - m_h^2}\right)$$

$$= \frac{1}{[\text{DEN}]_m} \left(1 - \frac{2k\cdot q}{k^2 - m_{(m)}^2}\right)\left(1 + \frac{2k\cdot p}{k^2 - m_h^2}\right)$$

$$\mathcal{M}^\mu = \frac{[\text{PRE}]}{[\text{DEN}]_m} \int d^4k \; \bar u(p') \Big[\big((\not k + \not q) + m_{(m)}\big) \gamma^\mu \big(\not k + m_{(m)}\big) \not k \Big] u(p)$$

$$\underbrace{\times \left(1 - \frac{2k\cdot q}{k^2 - m_{(m)}^2}\right)\left(1 + \frac{2k\cdot p}{k^2 - m_h^2}\right)}$$

$$\equiv \bar u(p') \,[\text{NUMERATOR}]\, u(p)$$

$$[\text{numerator}] = \underbrace{\left[(\not{k}+\not{q})+m_{(M)}\right]\gamma^{\mu}\left(\not{k}+m_{(M)}\right)\not{k}}\cdot\underbrace{\left(1-\frac{2k\cdot q}{k^2-m_{(M)}^2}\right)\left(1+\frac{2k\cdot p}{k^2-M_h^2}\right)}$$

$$\underbrace{\begin{aligned}&(\not{k}+\not{q})\gamma^{\mu}\not{k}\not{k} + (\not{k}+\not{q})\gamma^{\mu}m_{(M)}\not{k}\\ &+ m_{(M)}\gamma^{\mu}\not{k}\not{k} + m_{(M)}\gamma^{\mu}m_{(M)}\not{k}\end{aligned}}$$

$$\left(1+\frac{2k\cdot p}{k^2-M_h^2}-\frac{2k\cdot q}{k^2-m_{(M)}^2}\right)$$

$$=\left[1+2k\cdot\left(\frac{p}{k^2-M_h^2}-\frac{q}{k^2-m_{(M)}^2}\right)\right]$$

$$(\not{k}+\not{q})\gamma^{\mu}k^2 + \not{k}\gamma^{\mu}m_{(M)}\not{k} + \not{q}\gamma^{\mu}m_{(M)}\not{k}$$
$$+m_{(M)}\gamma^{\mu}k^2 + m_{(M)}^2\gamma^{\mu}\not{k}$$

$$\begin{aligned}= &\underset{①}{\not{k}\gamma^{\mu}k^2} + \underset{②}{\not{q}\gamma^{\mu}k^2} + \underset{③}{2m_{(M)}k^{\mu}\not{k}} - \underset{④}{m_{(M)}\gamma^{\mu}k^2}\\ &+\underset{⑤}{\not{q}\gamma^{\mu}m_{(M)}\not{k}} + \underset{⑥}{m_{(M)}\gamma^{\mu}k^2} + \underset{⑦}{m_{(M)}^2\gamma^{\mu}\not{k}}\end{aligned}$$

NOW USE: $k^{\alpha}k^{\beta} = \frac{1}{4}\eta^{\alpha\beta}k^2$

$p' = p+q \rightarrow q = p'-p$

LETS GO TERM BY TERM TO SEE WHAT THE CONTRIBUTION TO $(p+p')$ IS.

$$A = \frac{1}{k^2 - M_n^2} \qquad B = \frac{1}{k^2 - M_{(m)}^2}$$

drop

① $\cancel{k}\gamma^\mu k^2 \left[1 + 2k\cdot(Ap - Bq) \right]$ $\qquad p' = p+q$

$= \frac{1}{4}\cdot 2k^2 \,(A\not{p} - B\not{q})\gamma^\mu k^2$

$= \frac{1}{2} k^4 \,(A\not{p} - B\not{q})\gamma^\mu$

$= \frac{1}{2} k^4 \left[(A+B)\not{p} - B\not{p}' \right]\gamma^\mu$

↳ acts to right $= M_\mu \to$ drop.

$= \boxed{(A+B)k^4 p^\mu} + (\text{masses})\gamma^\mu$

② $\not{q}\gamma^\mu k^2 \left[1 + \underbrace{2k\cdot(Ap - Bq)}_{\to\, 0 \text{ by Lorentz}} \right]$

$= (\not{p}' - \not{p})\gamma^\mu k^2$

↳ gives m_e acting to left → drop

$= \boxed{-2p^\mu k^2}$

③ $2M_{(m)} k^\mu \cancel{k} \cdot 2k\cdot(Ap - Bq)$

$= 0$ by Lorentz, no $(p+p')^\mu$ contribution

④ $- M_{(m)}\gamma^\mu k^2 \cdot 2k\cdot(Ap - Bq)$

$= 0$ by Lorentz, no $(p+p')^\mu$ contribution

⑤ $\underbrace{\not{q}\,\gamma^\mu\, m_{(m)} \not{k}}\,[1 + 2k\cdot(Ap - Bq)]$

$(2q^\mu + \gamma^\mu \not{q})\, m_{(m)} \not{k}$ $\quad q = p' - p$

$$= 2q^\mu m_{(m)} \not{k} \times 2k\cdot(Ap - Bq) + \gamma^\mu \not{q}\, m_{(m)} \not{k} \times 2k\cdot(Ap - Bq)$$
$$= m_{(m)} k^2 q^\mu [(A+B)\not{p} - B\not{p}'] + \tfrac{1}{2}\gamma^\mu(\not{p}' - \not{p}) m_{(m)} k^2 (A\not{p} - B(\not{p}' - \not{p}))$$
$$= m_{(m)} k^2 q^\mu [(A+B)m_\mu - Bm_e] + \underbrace{\tfrac{1}{2} m_{(m)} k^2 \gamma^\mu (\not{p}' - m_\mu)[(A+B)m_\mu - B\not{p}']}$$

$$= m_{(m)} k^2 (A+B) m_\mu (p')^\mu$$
$$- m_{(m)} k^2 B m_\mu (p')^\mu$$

$$= m_{(m)} [(A+B)m_\mu - Bm_e]\, k^2 q^\mu$$
$$+ \boxed{m_{(m)} A m_\mu k^2 (p')^\mu}$$

note: q^μ term doesn't contribute to physical amplitude
remaining term is small ($\propto m_\mu$), but
let's keep it for now.

⑥ $m_{(m)} \gamma^\mu k^2 \times 2k\cdot(Ap - Bq)$

$= 0$ BY LORENTZ

⑦ $M_{(m)}^2 \gamma^\mu \not k \cdot 2k\cdot(Ap - Bq)$

$$= \frac{1}{2} M_{(m)}^2 \gamma^\mu k^2 \left(A\underset{m_\mu}{\not p} - B\underset{(\not p' - \not p)}{\not q}\right)$$

$\not p \to m_\mu$, $\quad \not q \hookrightarrow (\not p' - \not p)$, $\quad \not p \to m_\mu$

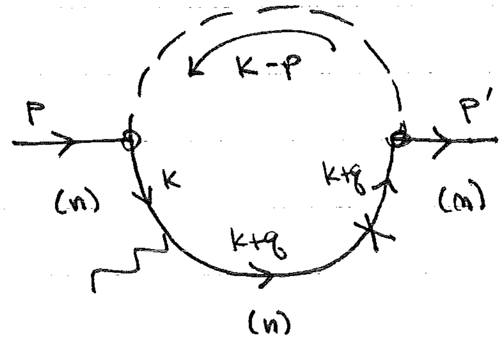
$$= -\frac{1}{2} M_{(m)}^2 k^2 B \gamma^\mu \not p'$$

$$= \boxed{- M_{(m)}^2 k^2 B (p')^\mu}$$

$$\boxed{\begin{aligned}[\text{NUMERATOR}] = \; & (A+B) k^4 p^\mu - 2p^\mu k^2 \\ & + A M_{(m)} m_\mu k^2 (p')^\mu - B M_{(m)}^2 k^2 (p')^\mu\end{aligned}}$$

<u>remark</u>: it is sufficient to just pick the p^μ coefficient to identify the gauge-invariant contribution.
WE'LL KEEP THE $(p')^\mu$ TERMS JUST FOR FUN.

$$\mathcal{M}^\mu = \int d^4k \, \frac{[\text{PRE}]}{[\text{DEN}]_m} \, \bar{u}(p') \, [\text{NUMERATOR}] \, u(p)$$



same prefactor

no mass term

$$\frac{1}{(k-p)^2 - M_h^2}$$

$$\mathcal{M}^{\mu} = [\mathrm{PRE}] \int d^4k \; \bar{u}(p') \left\{ \frac{\not{k}+\not{q}}{(k+q)^2 - M_{(M)}^2} \cdot \frac{\not{k}+\not{q}+M_{(n)}}{(k+q)^2 - M_{(n)}^2} \gamma^{\mu} \frac{\not{k}+M_{(n)}}{k^2 - M_{(n)}^2} \right\} u(p)$$

$$\left(\frac{1}{(k+q)^2 - M_{(M)}^2} \right) \frac{1}{(k+q)^2 - M_{(n)}^2} \; \frac{1}{(k-p)^2 - M_h^2} \; \frac{1}{k^2 - M_{(n)}^2}$$

$$\underbrace{\frac{1}{k^2 - M_{(M)}^2} \left(\frac{1}{k^2 - M_{(n)}^2} \right)^2 \frac{1}{k^2 - M_h^2}}_{\frac{1}{[\mathrm{DET}]_n}} \underbrace{\left(1 - \frac{2k\cdot q}{k^2 - M_{(M)}^2} \right) \left(1 - \frac{2k\cdot q}{k^2 - M_{(n)}^2} \right) \left(1 + \frac{2k\cdot p}{k^2 - M_h^2} \right)}$$

← note that this denominator is different from the previous one

$$\frac{1}{[\mathrm{DET}]_n} \left[1 + 2k \left(Ap - Bq - Cq \right) \right]$$

$$\frac{1}{k^2 - M_h^2} \qquad B = \frac{1}{k^2 - M_{(M)}^2} \qquad C = \frac{1}{k^2 - M_{(n)}^2}$$

$$[\text{numerator}] = \underbrace{(\not{k}+\not{q})\left[(\not{k}+\not{q})+m_{(n)}\right]}\gamma^\mu(\not{k}+m_{(n)})\left[1+2k(Ap\mp Bq-C_q)\right]$$

$$\left[(k+q)^2 + m_{(n)}(\not{k}+\not{q})\right]\gamma^\mu(\not{k}+m_{(n)})$$

$$(k+q)^2\gamma^\mu\not{k} \quad + \quad (k+q)^2 m_{(n)}\gamma^\mu$$
$$+\, m_{(n)}(\not{k}+\not{q})\gamma^\mu\not{k} \quad + \quad m_{(n)}^2(\not{k}+\not{q})\gamma^\mu$$

$$(k+q)^2\gamma^\mu\not{k} \quad + \quad (k+q)^2 m_{(n)}\gamma^\mu \quad + 2m_{(n)}k^\mu(\not{k}+\not{q})$$
$$-\, m_{(n)}(\not{k}+\not{q})\not{k}\gamma^\mu \quad + \quad m_{(n)}^2(\not{k}+\not{q})\gamma^\mu$$

$$\overset{①}{(k+q)^2\gamma^\mu\not{k}} \quad + \quad \overset{②}{(k+q)^2 m_{(n)}\gamma^\mu} \quad + \overset{③}{2m_{(n)}k^\mu(\not{k}+\not{q})}$$
$$\underset{④}{-\, m_{(n)}k^2\gamma^\mu} - \underset{⑤}{m_{(n)}\not{q}\not{k}\gamma^\mu} \quad + \underset{⑥}{m_{(n)}^2\not{k}\gamma^\mu} \quad \underset{⑦}{+\, m_{(n)}^2\not{q}\gamma^\mu}$$

$$① \ (k^2+2k\cdot q + \cancel{q^2}^{\,0})\gamma^\mu\overset{\not{k}}{\downarrow}\left[1+2k(Ap-Bq-C_q)\right]$$
$$= k^2\gamma^\mu\not{k}\cdot 2k(Ap-Bq-C_q)$$
$$+2k\cdot q\,\gamma^\mu\not{k}$$

$$= \tfrac{1}{2}k^4\gamma^\mu(A\underset{\substack{\downarrow\\ m_\mu}}{\not{p}}-B\not{q}-C\not{q}) \quad + \tfrac{1}{2}k^2\gamma^\mu\not{q}$$

$$= -\tfrac{1}{2}k^4\gamma^\mu(B+C)\not{q} \quad + \tfrac{1}{2}k^2\gamma^\mu\not{q}$$
$$= \boxed{\left(k^2-(B+C)k^4\right)(p')^\mu}$$

② $(k+q)^2 M_{(n)} \gamma^\mu \left[1 + 2k\cdot(Ap - Bq - Cq)\right]$

↳ no $(p+p')^\mu$ term

③ $2M_{(n)} k^\mu (\cancel{k} + \cancel{q})\ 2k\cdot(Ap - Bq - Cq)$

0 ↳ $(m_\mu + m_e)$

$$= \boxed{M_{(n)} (m_\mu + m_e) k^2 \left[Ap - (B+C)q\right]^\mu}$$

④ $-M_{(n)} k^2 \gamma^\mu \longrightarrow$ no $(p+p')$ term

Lorentz

⑤ $-M_{(n)} \cancel{q} \cancel{k} \gamma^\mu \left[\cancel{1} + 2k\cdot(Ap - Bq - Cq)\right]$

$-\frac{1}{2} k^2 M_{(n)} \cancel{q} (A\cancel{p} - B\cancel{q} - C\cancel{q}) \gamma^\mu$

$= -\frac{1}{2} M_{(n)} k^2 (A\cancel{q}\cancel{p} - (B+C)\cancel{q}^2) \gamma^\mu$

0

$= -M_{(n)} A k^2 \cancel{q}\, p^\mu$

$= \boxed{-M_{(n)} A k^2 (m_e - m_\mu) p^\mu}$ $\quad p' = p + q$

⑥ $m_{(n)}^2 \cancel{k} \gamma^\mu\, 2k\cdot(Ap - Bq - Cq)$

$= \frac{1}{2} m_{(n)}^2 (A\cancel{p} - (B+C)\cancel{q}) \gamma^\mu \quad k^2$

$= \boxed{m_{(n)}^2 (A+B+C) k^2 p^\mu}$

⑦ $m_{(n)}^2 \not{q} \gamma^\mu \left[1 + 2\cancelto{0}{k} \cdot (Ap - Bq - Cq') \right]$

$\boxed{= -2 m_{(n)}^2 p^\mu}$

$$
\begin{aligned}
[\text{NUMERATOR}] = \; & \left(k^2 - (B+C) k^4\right) (p')^\mu \\
& + m_{(n)} (M_e - M_n) k^2 \left[(A+B+C) p - (B+C) p' \right]^\mu \\
& - m_{(n)} (M_e - M_n) k^2 A p^\mu \\
& + m_{(n)}^2 (A+B+C) k^2 p^\mu \\
& - 2 m_{(n)}^2 p^\mu
\end{aligned}
$$

Review

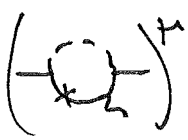
$$[\mathrm{PRE}] = Y^3 \frac{v}{\sqrt{2}}$$

$$\frac{1}{[\mathrm{DEN}]_m} = \left(\frac{1}{k^2 - m_{(m)}^2}\right)^2 \frac{1}{k^2 - m_{(n)}^2} \; \frac{1}{k^2 - m_h^2} = AB^2C$$

$$\frac{1}{[\mathrm{DEN}]_n} = \frac{1}{k^2 - m_{(m)}^2} \left(\frac{1}{k^2 - m_{(n)}^2}\right)^2 \frac{1}{k^2 - m_h^2} = ABC^2$$

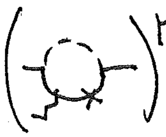
$$A = \frac{1}{k^2 - m_h^2} \qquad B = \frac{1}{k^2 - m_{(m)}^2} \qquad C = \frac{1}{k^2 - m_{(n)}^2}$$

M_1 $\qquad$ M_2



$$(\ldots)^\mu = \int d^4k \frac{[\mathrm{PRE}]}{[\mathrm{DEN}]_m} \bar{u}(p') \left[(A+B) k^4 p^\mu - 2k^2 p^\mu - B m_{(m)}^2 k^2 (p')^\mu \right] u$$

(DROPPING $M_{external}$ TERMS)



$$(\ldots)^\mu = \int d^4k \frac{[\mathrm{PRE}]}{[\mathrm{DEN}]_n} \bar{u}(p') \Big[k^2 (p')^\mu - (B+C) k^4 (p')^\mu$$

$$+ m_{(n)}^2 (A+B+C) k^2 p^\mu$$

$$- 2 m_{(n)}^2 p^\mu \Big] u(p)$$

[NOTE: TO GET $(p+p')$ COEFFICIENT, $p \to \frac{1}{2}(p+p') + \frac{1}{2}q$, $p' \to \frac{1}{2}(p+p') - \frac{1}{2}q$]

in Mathematica file: $m_h \to m$
$m_{(m)} \to M1$
$m_{(n)} \to M2$