ANNOUNCEMENTS

- MIDTERM DEBRIEFING ↗ topics · DRAW?
- FINAL EXAM ACCOMODATION
- VIDEO TAPING TODAY

(GRADING UPDATE)
- HONORS THESIS PRESENTATION
- SPS TALK, SHIRTS, YMUSA
- CYSTIC FIBROSIS 9:00 am (10)

AGENDA

- POTENTIALS & GAUGE TRANSFORMATIONS
- E&M FOR RETARDS :)
- PROBLEM SOLVING

WHERE WE ARE - the BIG PICTURE

- REVIEWED MAXWELL'S EQ - CRUX OF ALL OF E&M
  → EVERYTHING WE'VE DONE WE'VE USED MAXWELL'S EQ

$$\boxed{\begin{array}{ll} \nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\rho & \nabla\times\vec{E} = -\partial\vec{B}/\partial t \\ \nabla\cdot\vec{B} = 0 & \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\,\partial\vec{E}/\partial t \end{array}}$$

naively $-\nabla^2 V = \frac{1}{\epsilon_0}\rho$?
→ BUT $\nabla\times\nabla V \equiv 0$

WE'VE TALKED ABOUT WHAT THESE MEAN
SYMMETRY OF EQNS (MONOPOLES)

- E & $\vec{P}$ CONSERV. IN FIELDS (⇝ FIELDS ARE "REAL")
- EM WAVES : ME ⇒ WAVE EQ
- EM WAVES IN MEDIA: ME ⇒ BC ⇒ OPTICS, WAVEGUIDES
- NOW: new TOPIC ... GAUGE TRANSFORMATIONS
  → NOT AN APPLICATION OF ME
  → GOING BACK TO Q. OF POTENTIALS

What are potentials?

- CLASSICALLY THEY'RE NOT "REAL" (QM SAYS OTHERWISE)
- MATHEMATICAL TOOLS (from HELMHOLTZ THM, APPENDIX)
- THUS FAR WE'VE ONLY TALKED ABOUT ELECTRO/MAGNETO STATIC POTENTIALS
  → $\dot{\vec{E}}, \dot{\vec{B}} = 0$ so ∂_t TERMS IN ME DONT HAUNT US
  (thus is the game, right? in STUDYING EM WAVES in CONDUCTORS, $\vec{J}$ TERM HAUNTED US → $\vec{E}$)

in GENERAL : $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \neq 0 \Rightarrow \vec{E} \neq -\nabla V$

$\boxed{\nabla\cdot\vec{B} = 0} \Rightarrow \boxed{\vec{B} = \nabla\times\vec{A}}$ still (no ρ_m)

But a simple manipulation:

$$\nabla\times E = -\frac{\partial}{\partial t}(\nabla\times\vec{A}) \Rightarrow \nabla\times\left(E + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \boxed{(E + \partial\vec{A}/\partial t) = -\nabla V}$$

$$\boxed{\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}}$$

PLUG INTO OTHER MAXWELL EQ:

$$\left\{\boxed{\begin{aligned} &\nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\frac{1}{\epsilon_0}\rho \\ &\left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2 A}{\partial t^2}\right) - \nabla\left(\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}\end{aligned}}\right.$$

SO THESE ARE THE MAXWELL EQ FOR POTENTIALS

~~Note: 4 EQ, 4 UNKNOWNS~~

GAUGE TRANSFORMATIONS

SINCE POTENTIALS ARE "JUST TOOLS TO GET $\vec{E},\vec{B}$"
WE STILL HAVE SOME FREEDOM. GIVEN $(V,\vec{A})$ ← 4 COMP
WE CAN WRITE A NEW $(V',\vec{A}')$ THAT ⇒ SAME $(\vec{E},\vec{B})$ ← 6 COMP

$$\left.\begin{aligned}\vec{A}' &= \vec{A} + \vec{\alpha} \\ V' &= V + \beta\end{aligned}\right\} \text{ solve for } \vec{\alpha}, \beta$$

$\vec{B} = \nabla\times\vec{A}' = \nabla\times\vec{A} \Rightarrow \boxed{\nabla\times\vec{\alpha} = 0} \Rightarrow \vec{\alpha} = \nabla\lambda$ ← ANY SCALAR FUNCTION OF (x,y,z,t)

$\vec{E} = -\nabla V - \partial\vec{A}/\partial t = -\nabla V' - \partial\vec{A}/\partial t \Rightarrow \boxed{\nabla\beta + \partial\vec{\alpha}/\partial t = 0} \Rightarrow \nabla(\beta + \dot{\lambda}) = 0$
$\Rightarrow \beta = -\dot{\lambda} + k(t)$
$= -\dot{\lambda}$

ABSORB $k(t)$ INTO λ SINCE λ IS ARBITRARY ANYWAY!

$$\Rightarrow \boxed{\begin{aligned}\vec{A}' &= \vec{A} + \nabla\lambda \\ V' &= V - \partial\lambda/\partial t\end{aligned}}$$ LEAVES $\vec{E},\vec{B}$ <u>INVARIANT</u>

We already chose gauges in electrostatics / magnetostatics
- CHOSE POTENTIAL st $V=0$ @ ∞
- in deriving $\vec{A}$, WE HAD

$$\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \underbrace{\nabla(\nabla\cdot\vec{A})}_{\text{chose } \nabla\cdot\vec{A}=0} - \nabla^2\vec{A} = \mu_0\vec{J}$$

PHYSICAL SIGNIFICANCE

NOETHER'S THM: SYMMETRY (of L) ⇒ CONSERVATION LAW (8!)
→ pf. IS A LITTLE FORMAL FOR OUR CLASS (CLASSICAL FIELD THY)

NICE GAUGES - (not how to talk to girls)

$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}', t)}{r} d\tau'$

COULOMB : $\boxed{\nabla \cdot A = 0}$ $\Rightarrow$ $\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$ (POISSON EQ) EASY!

→ V DET. BY $\rho(\vec{r}, t = \text{NOW})$
→ WHAT ABOUT CAUSALITY? See Am J. Phy **35** 832
→ BUILT IN (we'll touch on this later)

from ☺ WE SEE THAT $\vec{A}$ IS UGLY TO CALCULATE though V IS GIVEN BY POISSON EQ.

LORENTZ $\boxed{\nabla \cdot A = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}}$

$$\Rightarrow \nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$
$$\Rightarrow \nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho$$

} inhomogeneous wave eq's

NOTICE : NICE SYMMETRY OF EQ'S
⇒ CLUE THAT $\vec{A}$ & V ARE RELATED INTIMATELY
i.e. 4-VECTOR IN RELATIVITY
⇒ JUST LIKE E & $\vec{p}$, or $\vec{S}$, $\overleftrightarrow{T}$

CALL $\Box = \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}$

ALSO CLUE OF STR : if $\mu_0 \varepsilon_0 = 1$ ⇔ $c = 1$
($\mu_0 \varepsilon_0 = 1$ IN NATURAL UNITS)
then looks like $\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2$

we will use this gauge from now on.

RETARDED STUFF - PLEASE GET YOUR GIGGLES OUT NOW

STATIC POTENTIALS

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau' \rightarrow$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau' \rightarrow$$

NONSTATIC (CLAIM)

$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}}\,d\tau' \\ \vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,', t_r)}{\mathscr{r}}\,d\tau' \end{aligned}}$$

$$\boxed{t_r = t - \mathscr{r}/c}$$

b/c EM TRAVELS @ SPEED OF LIGHT.

this isn't a PROOF, this is a guess!
TO PROVE IT, JUST SHOW IT SATISFIES

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{A} &= -\mu_0\varepsilon_0\frac{\partial V}{\partial t} \\ \Box^2 V &= -1/\varepsilon_0\,\rho \\ \Box^2\vec{A} &= -\mu_0\vec{J} \end{aligned}\right\}$$ LORENTZ GAUGE

→ note this is not true for fields
CANNOT JUST ADD IN RETARDED TIME!

→ GRIFFITHS PROVES THIS (PARTIALLY)
→ mostly math
→ just note in taking ∇ that ρ dep on $\vec{r}$ in t_r ALSO!

NOTE: COULD ALSO USE ADVANCED POTENTIALS $t_a = t + \mathscr{r}/c$
→ VIOLATES CAUSALITY
(THIS POPS UP A FEW TIMES IN PHYSICS)

ASSOCIATED FIELDS (JEFIMENKO EQ) $\frac{1}{\mathscr{r}}$ $\dot{\rho}$ $\dot{\vec{J}}$ ASSUME DEP. ON $\vec{r}\,'$, t_r

$$\boxed{\begin{aligned} E(\vec{r},t) &= \frac{1}{4\pi\varepsilon_0}\int \left[\frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}\,',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c^2\mathscr{r}}\right] d\tau' \\ \vec{B}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int \left[\frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c\mathscr{r}}\right]\times\hat{\mathscr{r}}\,d\tau' \end{aligned}}$$

Derivation is just churning the math

LIÉNARD-WIECHERT POTENTIALS

RETARDED POTENTIALS OF A POINT CHARGE ON PATH $\vec{w}(t)$

$$t_r = t - \frac{1}{c}\underbrace{|\vec{r} - \vec{w}(t_r)|}_{\mathscr{R}}$$

RETARDED POSITION

NOW USE RETARD. POTENTIAL $V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t_r)}{\mathscr{R}} d\tau'$

INDEP OF PRIMED COORDS

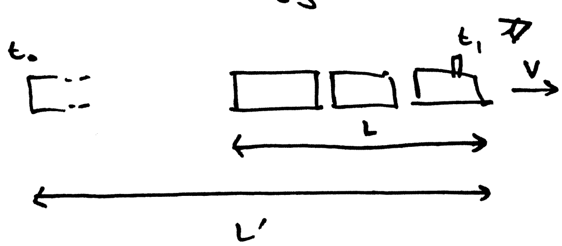
think: $\int \rho(\vec{r}', t_r)\, d\tau' = q \rightarrow$ WRONG!

MUST CALCULATE @ ONE INSTANT BUT $t_r = t - \mathscr{R}/c$ MEANS WE HAVE TO EFFECTIVELY CALC. USING ρ AT DIFFERENT TIMES!

(i.e. t_r dep. on SPACE)

<u>CLAIM</u>: $\int \rho(\vec{r}', t_r)\, d\tau' = \dfrac{q}{1 - \hat{\mathscr{R}}\cdot\vec{v}/c}$

train analogy:



t_0 : LIGHT FROM CABOOSE EMITTED that eventually reaches your eye @ t_1

t_1 : LIGHT FROM ENGINE reaches your eye as it passes you.

$$\Delta t = \frac{L'}{c} = \frac{L' - L}{v}$$

TIME for light to reach + from BACK

time for train engine to advance to your eye.

$$\Rightarrow L' = \frac{L}{1 - v/c}$$

(BECOMES $\frac{L}{1 - v\cos\theta/c}$)

LOOKS LIKE LORENTZ CONTR → BUT IT'S NOT!

· treat POINT CH. AS EXTENDED CH → take limit as SIZE → 0 (CORRECTION IS INDEP OF PARTICLE SIZE) so : $\tau' = \dfrac{\tau}{1-\hat{\mathscr{r}}\cdot\vec{v}/c}$

$$\Rightarrow \quad V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left(\frac{qc}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}\right)$$

$$A(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\rho(\vec{r}',t_r)\,v(t_r)}{\mathscr{r}}\,d\tau'$$

$$= \frac{\mu_0}{4\pi}\frac{\vec{v}}{\mathscr{r}}\int \rho(\vec{r}',t_r)\,d\tau'$$

$$= \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})}$$

LW POTENTIALS

NOW WE WANT THE "PHYSICAL" FIELDS $\vec{E}, \vec{B}$

$$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$$
$$\vec{B} = \nabla\times\vec{A}$$

$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$

$u = c\hat{\mathscr{r}} - \vec{v}$

gives E.S. case

$$V, \vec{A} \longrightarrow \boxed{\text{MAGIC BOX OF UGLY MATH!}} \longrightarrow \begin{cases} \vec{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\Big[\underbrace{(c^2-v^2)\vec{u}}_{\mathrm{I}} + \underbrace{\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})}_{\mathrm{II}}\Big] \\ \vec{B} = \dfrac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t) \end{cases}$$

$\dot{\vec{v}}$

FAINT OF HEART

$\vec{B}\perp\vec{E}$

$\downarrow$ RETARDED $\vec{\mathscr{r}}$

(I) : $\rightarrow$ ELECTROSTATIC RESULT

(II) : $\rightarrow$ FALLS OFF AS $1/\mathscr{r}$ $\rightarrow$ DOMINATES @ LARGE DIST

$\Rightarrow$ RADIATION (ACCEL $\rightarrow$ FALLS OFF SLOWLY)

EXCERSISES - DIVIDE BY DRAW #'s!

GRIF. 10.3. FIND $\vec{E}, \vec{B}, \rho, \vec{J}$ FOR

GRP 1 → $V(\vec{r},t) = 0 \qquad \vec{A}(\vec{r},t) = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r^2}\hat{r}$

GRP 2 →

10.5. USE $\lambda = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r}$ ON ↑

$$E = -\nabla V - \frac{\partial \vec{A}}{\partial t} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

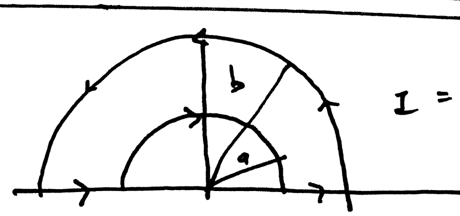
$$B = \nabla \times \vec{A} = 0$$

→ STATIONARY PT CHARGE!

$$V' = V - \frac{\partial \lambda}{\partial t} = 0 - \left(-\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right)$$

$$\vec{A}' = \vec{A} + \nabla\lambda = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r^2}\hat{r} + -\frac{1}{4\pi\varepsilon_0}qt\left(-\frac{1}{r^2}\hat{r}\right) = 0$$

~~[illegible]~~



$I = kt$

10.15

→ CALC RETARDED $\vec{A}$ AT CENTER ↙

→ ? $\vec{E}$ AT CENTER (WHY $\exists$ $\vec{E}$?)

→ ? $\vec{B}$? (CAN'T COMPUTE) → $\nabla\times\vec{A}$

~~BEFORE~~
dep of A, E on t?
dep of A, E on k
DIR OF $\vec{A}, \vec{E}$?
@ CENTER

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{I}(t_r)}{r}\,dl$$

$$= \frac{\mu_0 k}{4\pi}\int\frac{(t - r/c)}{r}\,d\vec{l} = \frac{\mu_0 k}{4\pi}\left\{t\int\frac{d\vec{l}}{r} - \frac{1}{c}\cancelto{0}{\oint d\vec{l}}\right\}$$

$$= \frac{\mu_0 k t}{4\pi}\left\{\underbrace{\frac{1}{a}\int_1 d\vec{l}}_{2a\hat{x}} + \underbrace{\frac{1}{b}\int_2 d\vec{l}}_{-2b\hat{x}} + 2\hat{x}\int_a^b\frac{dx}{x}\right\}$$

$$\vec{A} = \frac{\mu_0 k t}{4\pi}\left[\frac{1}{a}(2a) + \frac{1}{b}(-2b) + 2\ln(b/a)\right]\hat{x}$$

$$= \frac{\mu_0 k t}{2\pi}\ln(b/a)\,\hat{x}$$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0 k}{2\pi}\ln b/a\,\hat{x}$$

CUTE EXAMPLE 10.4 (qualitative)

POINT CHARGE w/ CONST VELOCITY

$\Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{(c^2 - v^2)\,\mathscr{r}}{(\vec{\mathscr{r}} \cdot \vec{u})^3}\, \vec{u}$ $\qquad \vec{u} = c\hat{\mathscr{r}} - \vec{v}$

~~[illegible]~~

$\downarrow$

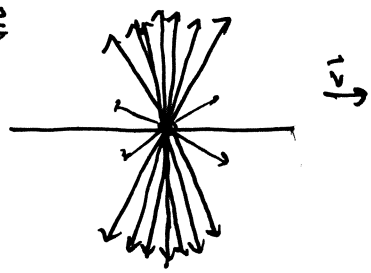
$\boxed{\text{Math}}$

$\downarrow$

from present position (COINCIDENCE)

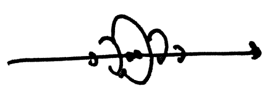
$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2\theta / c^2)^{3/2}} \frac{\hat{R}}{R^2}$$

$\vec{E}$



LORENTZ CONTRACTION! (looks like it, at least)

$\vec{B}$



WHAT YOU'D EXPECT.

RADIATION : POWER THAT FLOWS TO ∞

$$P(r) = \oint \vec{S}\cdot d\vec{a} = \frac{1}{\mu_0}\oint(\vec{E}\times\vec{B})\, d\vec{a}$$

AREA $\propto r^2$

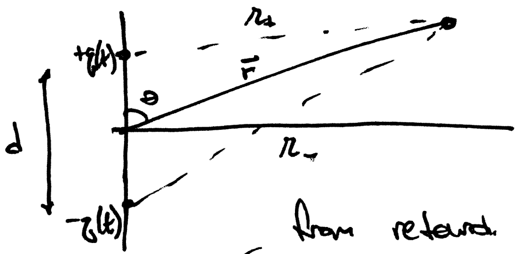
$\Rightarrow \vec{S}$ FALLS NO FASTER THAN $1/r^2$

i.e. $\vec{E}$ & $\vec{B}$ CAN EACH GO LIKE $1/r$

$\Rightarrow$ time-dep. fields in JEFIMENKO's Eq!

ELECTRIC DIPOLE $\rightarrow$ SMALL SEPARATION, BIG OBS. DIST

3 SCALES: λ, d, r



$$q(t) = q_0 \cos(\omega t)$$

DIPOLE : $\vec{p}(t) = \underset{q_0 d}{\underset{\uparrow}{p_0}} \cos(\omega t)\,\hat{z}$

from retard. pot.

$$\Rightarrow V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q_0\cos[\omega(t - r_+/c)]}{r_+} - \frac{q_0\cos[\omega(t-r_-/c)]}{r_-}\right\}$$

$$r_\pm = \sqrt{r^2 \mp r d\cos\theta + (d/2)^2} \qquad \text{(LAW OF COS)}$$

APPROX 1: $d \ll r \Rightarrow r_\pm \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right)$

$$\uparrow \sqrt{1\mp\varepsilon} = 1 \mp \varepsilon$$

$$\Rightarrow \frac{1}{r_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\cos[\omega(t - r_\pm/c)] \cong \cos\left[\omega(t - r/c) \pm \frac{\omega d}{2c}\cos\theta\right]$$

$$= \cos[\omega(t-r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin[\omega(t-r/c)]\sin\left(\frac{\omega d}{2c}\cos\theta\right)$$

APPROX 2 : $d \ll c/\omega$ $\left(\Leftrightarrow d \ll \lambda \quad \lambda = 2\pi c/\omega\right)$

$$\cos[\omega(t - r_\pm/c)] \approx \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c}\cos\theta\,\sin[\omega(t-r/c)]$$

$$\Rightarrow V = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)] + \underbrace{\frac{1}{r}\cos[\omega(t-r/c)]}_{\text{STATIONARY DIPOLE}}\right\}$$

APPROX 3: $r \gg c/\omega$ $(r \gg \lambda)$ KILLS STATIONARY PT.

$$V \to \boxed{-\frac{P_0 \omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)]}$$

VECTOR POTENTIAL

$$I(t) = -q_0 \omega \sin(\omega t)\,\hat{z}$$

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{1}{r}\left[-q_0 \omega \sin[\omega(t - r/c)]\,\hat{z}\right] dz$$

$$\boxed{\vec{A} = -\frac{\mu_0 P_0 \omega}{4\pi} \sin[\omega(t - r/c)]\,\hat{z}}$$

MATH gets $\vec{E}, \vec{B}$ (drop terms accordingly)

$$\vec{S} = \frac{\mu_0}{c}\left\{\frac{P_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{r}$$

no rad in dir of dipole axis

INTENSITY IS AVG OVER CYCLE (ie over t)

$$\langle S \rangle = \left(\frac{\mu_0 P_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{r}$$

COULD WE HAVE GUESSED THIS?

$V \propto P_0 \omega \sin(\omega(t - r/c))$ ($\frac{1}{2}$ DISTANCE $2\times\omega$ → SAME STRENGTH ?)

A SHOULD ALSO FOLLOW

$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ → PULLS OUT ω → $\vec{E} \propto P_0 \omega^2$

$S \propto \vec{E} \times \vec{B} \propto P_0^2 \omega^4$

$\sin^2\theta$ SINCE $\vec{E} \propto \sin\theta$

$\frac{1}{r^2}$ FROM AREA OF SPHERE