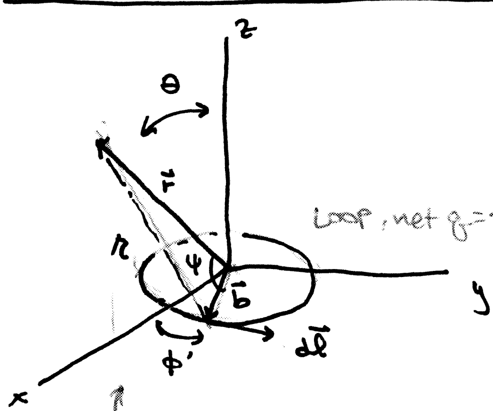
AGENDA

- ANNOUNCEMENTS: HW #3 / MIDTERM GRADING ASAP , YUEN
- MAGNETIC DIPOLE RADIATION (brief)
- POINT CHARGES
- RELATIVITY PRIMER
- PROBLEMS

MAGNETIC DIPOLE RADIATION



By symm can take
$\vec{r} = r\sin\theta\,\hat{x} + r\cos\theta\,\hat{z}$
$b = b\cos\phi'\,\hat{x} + b\sin\phi'\,\hat{y}$

$$I(t) = I_0\cos(\omega t)$$

$$\vec{m}(t) = \pi b^2 I(t)\,\hat{z} = m_0\cos(\omega t)\,\hat{z}$$

$m_0 = \pi b^2 I_0$

MAGNETIC DIPOLE

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{I_0\cos[\underbrace{\omega(t-\mathscr{r}/c)}_{\omega t_r}]}{\mathscr{r}}\,d\vec{l}$$

SYMMETRY

$$\boxed{\vec{A}(\vec{r},t) = \underbrace{\frac{\mu_0 I_0 b}{4\pi}}_{k}\,\hat{y}\int_0^{2\pi}\frac{\cos[\overbrace{\omega(t-\mathscr{r}/c)}^{\omega t_r}]}{\mathscr{r}}}$$

$\cos\phi'\,d\phi'$

(-1) → DETERMINE APPROXIMATIONS

$b \ll r$
$b \ll \lambda$
$r \gg \lambda$

$\lambda = 2\pi\frac{c}{\omega}$ (dimens)

(0) DETERMINE $\vec{A}$ USING

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r},t_r)}{\mathscr{r}}\,d\tau'$$

(1) DETERMINE $\mathscr{r}$ USING LAW OF COSINES

$$\mathscr{r}^2 = r^2 + b^2 - \underbrace{2rb\cos\psi}$$

$$= 2\vec{r}\cdot\vec{b} = rb\sin\theta\cos\phi'$$

(2) LOOP SMALLER THAN OBS. DISTANCE ($\frac{b}{r}$ is small) } DIM. LESS

$$\mathscr{r}^2 \cong r^2\left(1 - \frac{2b}{r}\sin\theta\cos\phi'\right) \leftarrow \text{DROP } O\left(\left(\tfrac{b}{r}\right)^2\right)$$

$$\mathscr{r} \cong r\sqrt{1-\epsilon} \approx r\left(1-\tfrac{1}{2}\epsilon\right) \leftarrow \boxed{\epsilon = \frac{2b}{r}\sin\theta\cos\phi'}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\cdot\frac{1}{1-\epsilon'} \approx \frac{1}{r}(1+\epsilon')$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi'\right)$$

$$= \frac{1}{r}\left(1 + \tfrac{1}{2}\epsilon\right)$$

(2a) ALSO APPROXIMATE $\cos(\omega t_r)$

$$\cos[\omega(t - r/c)] \cong \cos[\omega(t - \tfrac{1}{c}\{r(1 - \tfrac{1}{2}\epsilon)\})]$$

$$\cos[\omega(t - \tfrac{r}{c}) + \tfrac{\omega}{2c}\epsilon]$$

CALL THIS $\omega\tilde{t}_r$

(2b') USE $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\cos(\omega t_r) \approx \cos[\underbrace{\omega(t - r/c)}_{\omega\tilde{t}_r}]\cos[\tfrac{\omega\epsilon}{2c}] - \sin[\underbrace{\omega(t - r/c)}_{\omega\tilde{t}_r}]\sin[\omega\epsilon/2c]$$

(3) SIZE OF DIPOLE SMALL VS. WAVELENGTH

note: $\epsilon = \frac{2b}{r}\sin\theta\cos\phi'$

so $\frac{\omega\epsilon}{2c} \propto b/(c/\omega)$

$\cos\omega\epsilon/2c \to 1$

$\sin\omega\epsilon/2c \to \omega\epsilon/2c$

$$\cos(\omega t_r) \approx \cos[\underbrace{\omega(t - r/c)}_{\omega\tilde{t}_r}] - \overbrace{\tfrac{\omega b}{c}\sin\theta\cos\phi}^{\frac{\omega\epsilon r}{2c}}\sin[\underbrace{\omega(t - r/c)}_{\omega\tilde{t}_r}]$$

(4) PLUG INTO $\vec{A}(\vec{r}, t) = k\hat{y}\int \cos(\omega t_r)/\imath \,\cos\phi'\,d\phi'$

$$= k\hat{y}\int\{\cos[\underbrace{\omega(t - r/c)}_{\omega\tilde{t}_r}] - \tfrac{\omega\epsilon r}{2c}\sin[\underbrace{\omega(t - r/c)}_{\omega\tilde{t}_r}]\}\tfrac{1}{r}(1 + \tfrac{1}{2}\epsilon)\cos\phi'\,d\phi'$$

$$\approx \frac{k}{r}\hat{y}\int\{\cos(\omega\tilde{t}_r) + \tfrac{1}{2}\epsilon\cos(\omega\tilde{t}_r) - \tfrac{\omega\epsilon r}{2c}\sin(\omega\tilde{t}_r)\}\cos\phi'\,d\phi'$$

INTEG. TO ZERO

HAS ϕ' dependence

$$\approx \frac{k}{r}\hat{y}\left(\frac{b}{r}\sin\theta\right)\int \cos^2\phi'\left(\cos(\omega\tilde{t}_r) - \tfrac{\omega r}{c}\sin(\omega\tilde{t}_r)\right)d\phi'$$

INTEGRATES TO π

$\Rightarrow$ turn $\hat{y} \to \hat{\phi}$

$$\hookrightarrow \vec{A} = \frac{\pi k b}{r^2}\sin\theta\left(\cos(\omega\tilde{t}_r) - \tfrac{\omega r}{c}\sin(\omega\tilde{t}_r)\right)\hat{\phi}$$

(5) $r \gg c/\omega$ in RADIATION ZONE

so $\frac{c}{r\omega} \ll 1 \Rightarrow \cos(\omega t_r) \ll \frac{\omega r}{c}\sin(\omega t_r)$

$$\Rightarrow A(r,\theta,t) = -\frac{kb}{r^2}\sin\theta \, \frac{\omega r}{c}\sin(\omega t_r)\,\hat{\phi}$$

$$= \boxed{-\frac{\mu_0 I_0 b^2}{4\pi c}\,\frac{\sin\theta}{r}\,\sin(\omega(t - r/c))\,\hat{\phi}}$$

$\boxed{\text{find}}$:

$$\vec{E} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\phi}$$

$$\vec{B} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\theta}$$

ALWAYS TRUE

- → in PHASE, $\perp$, TRANSVERSE TO PROPAGATION $\hat{r}$
- → $E_0/B_0 = c$
- → LOOK JUST LIKE E. DIPOLE w/ $m_0 \leftrightarrow cp_0$, $(\hat{\theta},\hat{\phi}) \leftrightarrow (\hat{\phi}, -\hat{\theta})$

$$\langle \vec{S} \rangle = \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\,\frac{\sin^2\theta}{r^2}\,\hat{r}$$

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

$$P_M/P_E = \left(\frac{m_0}{p_0 c}\right)^2$$

ARBITRARY SOURCE RADIATION

- THIS MAY BE THE PINNACLE OF UGLY MATH IN THE BOOK
- YOU SHOULD GO OVER THIS. IT WOULD TAKE US FOREVER :(
- CUTE TO NOTE: USE OF CONS. OF CHARGE
  → READ HIS COMMENTS ON P. 418 FIRST

IDEA: MULTIPOLE EXPANSION

IF ELEC. 2^n-POLE VANISHES (OR $\frac{d^2}{dt^2}$ is ZERO)
MUST LOOK @ NEXT TERM, $(n+1)$ @ in r'
→ WHICH IS RELATED TO $\begin{cases} \text{MAGNETIC } 2^n\text{-POLE} \\ \text{ELECTRIC } 2^{(n+1)} \text{ POLE} \end{cases}$

POINT CHARGES

from last time:

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$

$$\uparrow \quad \vec{u} = c\hat{\mathscr{r}} - \vec{v}$$

$$\vec{B} = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t)$$

$$\vec{S} = \frac{1}{\mu_0 c}\left[E^2\hat{\mathscr{r}} - (\hat{\mathscr{r}}\cdot\vec{E})\vec{E}\right]$$

terrible analogy: college students

RECALL ARGUMENT:
RADIATION GOES TO ∞ (flies that leave garbage truck)
SURFACE AREA $\propto \mathscr{r}^2$
SO E. FLUX MUST GO AS $1/\mathscr{r}^2$ OR $\underbrace{\text{MORE SLOWLY}}_{\text{UNPHYSICAL}}$

$\vec{S} \propto E^2 \Rightarrow$ PART OF $\vec{E}$ that GOES AS $1/\mathscr{r}$ IS RADIATION

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$

FOR CHARGE @ REST @ $t = t_r$, $\vec{u} = c\hat{\mathscr{r}}$

$$\vec{E}_{rad} = \frac{\mu_0 q}{4\pi\mathscr{r}}\left[(\hat{\mathscr{r}}\cdot\vec{a})\hat{\mathscr{r}} - \vec{a}\right] \leftarrow \frac{q}{4\pi\epsilon_0 c\mathscr{r}}\left[\hat{\mathscr{r}}\times(\hat{\mathscr{r}}\times\vec{a})\right] = \cancel{\frac{\mu_0 q}{4\pi\mathscr{r}}}$$

$$\Rightarrow \vec{S}_{rad} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\mathscr{r}} \leftarrow \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi\mathscr{r}}\right)^2\left[a^2 - (\hat{\mathscr{r}}\cdot\vec{a})^2\right]\hat{\mathscr{r}}$$

$$\Rightarrow P = \boxed{\frac{\mu_0 q^2 a^2}{6\pi c}} \longrightarrow \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$$

holds for $v \ll c$ — cap cons eq. — Liénard's gen for $\vec{v}$ general

RADIATION REACTION

ACCELERATING CHARGE → RADIATION

· E CARRIED AWAY @ EXPENSE OF KE (!)

⇒ APPARENTLY RADIATION EXERTS RECOIL FORCE!

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \xrightarrow{\text{SUGGESTS}} \vec{F}_{rad} \cdot \vec{v} = -\frac{\mu_0 q^2 a^2}{6\pi c}$$

↑ $\frac{d}{dt}(\text{force} \cdot \text{dist})$

<u>BUT</u>: ONLY ACCOUNTS FOR RADIATION FIELDS

RADIATION FIELDS

PARTICLE

VELOCITY FIELDS

* (so our sugg. is correct when integ over full cycles (i.e. vel field E goes back & forth))

TO CALC. RECOIL FORCE, WE <u>REALLY</u> NEED <u>TOTAL POWER LOST</u> AT AN <u>INSTANT</u> (not just tot power that → ∞)

$$\left(\mathcal{E} \propto E^2 = \underbrace{E_v^2 + 2\vec{E}_v \cdot \vec{E}_a}_{\text{"velocity"}} + E_a^2 \right)$$

$$* \rightarrow \int_{t_1}^{t_2} \vec{F}_{rad} \cdot \vec{v}\, dt = -\frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} a^2\, dt$$

$$\frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt}\, dt = \cancel{\left(v \frac{d\vec{v}}{dt}\right)\Big|_{t_1}^{t_2}} - \int \frac{d^2\vec{v}}{dt^2} \cdot \vec{v}\, dt$$

= 0 BY CYCLES

⇓

$$\int_{t_1}^{t_2} \left(\vec{F}_{rad} - \frac{\mu_0 q^2}{6\pi c} \dot{\vec{a}} \right) \cdot \vec{v}\, dt = 0$$

$$\Rightarrow \boxed{\vec{F}_{rad} = \frac{\mu_0 q^2}{6\pi c} \dot{\vec{a}}}$$

not a pf!
the avg of $\parallel \vec{v}$ component

STR & WHY WE CARE

· MANY HINTS SO FAR THAT STR BUILT INTO E & M
→ SPEED OF LIGHT
EFFECTIVE LORENTZ CONTRACTION
even RETARDED POTENTIAL

eg.

+ − + − + −
+ − + − + −

+→ ← +→ ← +→ ←
+→ ← +→ ← +→ ←

⊕ move right
⊖ move left

LAB FRAME: WIRES ATTRACT
(expl: B-field)

IN rest frame of a − CHARGE IN WIRE 1

⇒ lorentz contraction of + CHARGES IN 2

− ⊖ −

⊖ ⊖ ⊖
⊕⊕⊕⊕⊕⊕⊕

net ELECTRIC FORCE

→ AWESOME.

VELOCITY

VELOCITY IS A TRICKY THING IN STR SINCE

$$V = \frac{d(\text{POSITION})}{d(\text{TIME})}$$

BUT WHICH POSITION & TIME DO WE USE? (which frame)

MOTIVATION: WANT TO USE A VELOCITY THAT IS A 4-VECTOR!

SO CAN USE dx^μ IN NUMERATOR
→ DENOMINATOR SHOULD BE AN INVARIANT (ie DOES NOT CHANGE UNDER Λ)

why? B/c. dx^μ ALREADY HAS THE CORRECT TRANSF.

A GOOD INVARIANT IS THE PROPER TIME $d\tau$
ie. THE TIME YOU MEASURE IN THE OBJECT'S REST FRAME

PROPER VELOCITY $\boxed{\eta^\mu = \frac{dx^\mu}{d\tau}}$ $\quad \eta^0 = \frac{c\,dt}{d\tau} = \frac{c}{\sqrt{1-u^2/c^2}}$

$\vec{\eta} = \frac{1}{\sqrt{1-u^2/c^2}}\,\vec{u}$ for "ORDINARY VELOCITY" $\vec{u}$

MAKES SENSE, W.r.t SINGLE OBS.

η TRANSF. AS A 4 VECTOR: $\bar{\eta}^\mu = \Lambda^\mu_\nu \eta^\nu$
$\vec{u}$ TRANSFORMS LIKE A TURD.

Q: WHAT IS $(\eta^\mu)^2$? c^2 ; WE MOVE IN SPACETIME @ SPD OF LIGHT

MOMENTUM

AGAIN, WANT A 4-VECTOR
NATURALLY $\boxed{p^\mu = m\eta^\mu}$

$$p^0 = \frac{mc}{\sqrt{1-u^2/c^2}}$$

IN NATURAL UNITS, $p^\mu = (E, \vec{p})$ $\quad (c=1)$

$P \cdot P = p^\mu p_\mu$ IS A SCALAR (INVARIANT)

→ $\boxed{E^2 - p^2c^2 = m^2c^4}$ ← how do you get $E = mc^2$?

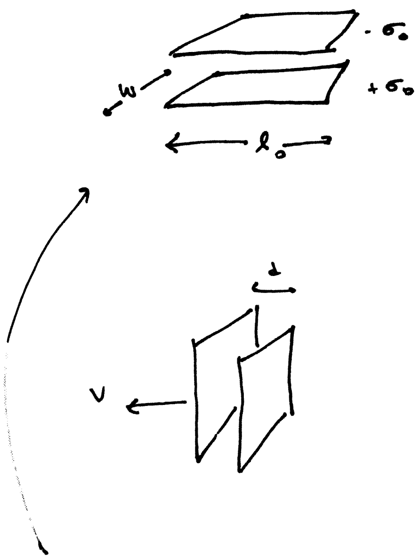
CALCULATE @ REST
& MUST BE THE SAME
IN ALL FRAMES

HOW FIELDS TRANSFORM

- ASSUME CHARGE IS INVARIANT (KINDA MAKES YOU WONDER WHERE CH. CAME FROM ... GAUGE FREEDOM)

- APPROACH: STUDY TRANSF OF CONST $\vec{E}$ & $\vec{B}$ FIELDS & TRY TO GENERALIZE

CAPACITOR



$$\vec{E}_0 = \frac{\sigma_0}{\varepsilon_0}\hat{y}$$

GIVE PLATES VELOCITY v GOING LEFT

$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{y}$ ONLY DIFF IS σ!

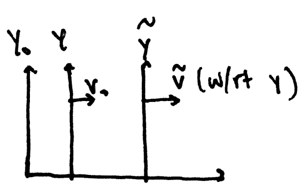
WHY? GAUSS' LAW

$$\sigma = \gamma_0\sigma_0$$
$$\vec{E}^{\perp} = \gamma_0\vec{E}_0^{\perp}$$

for PARALLEL COMPONENT $E^{\parallel}$ INDEP OF d IN CONFIG LEFT.

$$\Rightarrow E^{\parallel} = E_0^{\parallel}$$

$$\vec{K}_{\pm} = \mp\sigma v_0\hat{x} \Rightarrow \begin{matrix} B_z = -\mu_0\sigma v_0 \\ \tilde{B}_z = -\mu_0\tilde{\sigma}\tilde{v} \end{matrix}$$



$$\tilde{v} = \frac{v + v_0}{1 + vv_0/c^2}$$

$$\tilde{\sigma} = \tilde{\gamma}\sigma_0$$

$\tilde{n} = \gamma n$ & so forth.

Math $\Rightarrow$

$$\tilde{E}_x = E_x \qquad \tilde{E}_y = \gamma(E_y - vB_z) \qquad \tilde{E}_z = \gamma(E_z + vB_y)$$
$$\tilde{B}_x = B_x \qquad \tilde{B}_y = \gamma\left(B_y - \frac{v}{c^2}E_z\right) \qquad \tilde{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

WE NOTICE THIS HAS THE SAME TRANSFORMATION PROPERTIES AS A RANK 2 ANTISYMMETRIC TENSOR

$$\hookrightarrow t^{\mu\nu} = -t^{\nu\mu}$$

$$t^{\mu\nu} \rightarrow \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\,t^{\alpha\beta}$$

RELATIVISTIC E & M

4-VECTOR CURRENT DENSITY: $\boxed{J^\mu = \rho_0 \eta^\mu} = (c\rho, \vec{J})$

CONTINUITY EQ: $\nabla \cdot \vec{J} = -\dot{\rho} \Leftrightarrow \boxed{\partial_\mu J^\mu = 0}$

Maxwell: $\boxed{\partial_\nu F^{\mu\nu} = \mu_0 J^\mu, \quad \partial_\nu G^{\mu\nu} = 0}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & & (\vec{E}/c) & \\ & 0 & B_z & -B_y \\ -\left(\frac{\vec{E}}{c}\right) & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{pmatrix}$$

$G^{\mu\nu} = F^{\mu\nu}$ w/ $\frac{\vec{E}}{c} \to \vec{B}$, $\vec{B} \to -\vec{E}/c$

FORCE LAW

MINKOWSKI FORCE $\boxed{K^\mu = q\eta_\nu F^{\mu\nu}}$

$$\frac{dp^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dp^\mu}{dt} \Rightarrow \vec{K} = \frac{1}{\sqrt{1-u^2/c^2}}\vec{F}$$

A BETTER FORMULATION: POTENTIALS

$$A^\mu = \left(\frac{V}{c}, \vec{A}\right)$$

$$F^{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu$$

SIMPLEST ANTISYMMETRIC 2 TENSOR WE COULD MAKE OUT OF DERIVATIVES OF A

MAXWELL: $\boxed{\Box^2 A^\mu = -\mu_0 J^\mu}$ (LORENTZ GAUGE)

IF WE WERE CLEVER, WE COULD HAVE GUESSED THIS!

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t} \end{aligned} \Rightarrow \begin{aligned} B_z &= \partial_1 A^2 - \partial_2 A^1 \\ \tfrac{1}{c}E_z &= \partial_3 A^0 - \partial_0 A^3 \end{aligned}$$