SOLUTIONS

Assignment 4

Due date: Wednesday, February 20

1. H&F 4.10
2. H&F 4.11
3. H&F 4.12
4. H&F 4.13
5. Derive the equations of motion for the polymer chain model described in class but for the case of $N = 3$ mass points. Use the method of Lagrange multipliers and, by solving for them explicitly, eliminate them from the equations of motion. You do not have to solve the equations of motion.

FIGURE 4.18

c) A tunnel is bored through the Earth in a straight line that passes within a distance D from the center. Let x be the distance along this tunnel from the tunnel center as shown in Figure 4.18. When $x = 0, r = D$. *Using Lagrangian mechanics* (*not vectors*!), with x as the generalized coordinate, find the equation of motion for a particle of mass m dropped into the tunnel. The particle moves under gravity without friction or air resistance. Show that the motion is simple harmonic motion and that the period is the same as that of the satellite in part a) above. If the particle is dropped into the tunnel from the surface at the moment the satellite passes overhead, it will return to the starting point just as the satellite completes one orbit of the Earth.

d) A sloppy physicist known by the initials S. P. makes the following incorrect derivation of $V(r)$ in part b) above: "Let $M(r)$ stand for the part of the Earth's mass inside a sphere of radius $r \le R_e$. Then $M(r) = M_e(\frac{r}{R_e})^3$, so, for $r \le R_e$,

$$V_{SP}(r) = -\frac{GmM(r)}{r}, \tag{4.95}$$

since the matter outside a sphere of radius r does not contribute to the gravitational potential." What equation of motion will S. P. get for part b), and what is wrong with this reasoning?

Center of Mass Motion

Problem 10: *(Two massive bodies in a constant field)* Two massive bodies move in a constant external gravitational field. Show that their motion can be reduced to an equivalent one-body problem, just as we did for the Kepler problem.

Problem 11: *(N-body system)* Prove that, for an N-body system, if the external force on the ith body is $\vec{F}_i$, the center of mass moves like a single particle of mass $M = M_1 + \ldots M_N$ acted upon by a force $\vec{F} = \vec{F}_1 + F_2 + \ldots \vec{F}_N$. Use Lagrangian methods to prove this and to prove that we can ignore internal forces as far as center of mass motion is concerned.

Problem 12*: *(Explosion of projectile)* A projectile in outer space subjected to no external forces suddenly explodes into three pieces, which have Cartesian coordinates $\vec{r}_1, \vec{r}_2, \vec{r}_3$ with respect to an inertial reference frame. The three masses are m_1, m_2, and m_3. Assume that the forces during the explosion are not known, but it is believed that they can be derived from potentials depending only on the distances between pairs of the particles:

$$V = V_A(|\vec{r}_1 - \vec{r}_2|) + V_B(|\vec{r}_2 - \vec{r}_3|) + V_C(|\vec{r}_3 - \vec{r}_1|). \tag{4.96}$$

a) Show that the center of mass moves at the same constant velocity it had before the explosion.

b) Show that, in the center of mass reference frame, the three fragments lie in a plane after the explosion. Hint: Prove that the total momentum in this reference frame is zero.

c) Derive what happens to the center of mass after the explosion if instead of being "external force-free," the system also had a constant gravitational force on it. Does the result depend on the force being a constant? Is the total momentum of the fragments still zero in the noninertial center of mass reference frame?

~~Central Force~~ Problems

Problem 13: *(Massive particle moving on a cone)* A massive particle moves under the ~~acceleration~~ of gravity, g, and without friction on the surface of a cone of revolution with half angle α. Find the Lagrangian in plane polar coordinates. Also find the equation of motion for r and the effective potential $V_{\text{eff}}(r)$. If the particle is launched horizontally with velocity v_0 at a height z_0, prove that the condition for circular motion is $v_0^2 = gz_0$.

Problem 14*: *(Two connected masses)* Two masses m_1 and m_2 are connected by a weightless string of fixed total length l_0. Mass m_1 rests on a frictionless table, which has a small hole cut into it. Mass m_2 hangs down vertically from this hole. Assume that m_2 can only move in the vertical direction, so the problem has two degrees of freedom.

a) Assuming that the acceleration of gravity is g, find the Lagrangian and the equations of motion for this system. (Use plane polar coordinates.)

b) The total energy $E = T + V$ is a constant of the motion. How can you see this by inspection of the Lagrangian? There is a second constant of the motion. Explain how to find it, and prove that it is indeed constant. (Call this constant l). What is the physical interpretation of l?

c) Is there a case where the motion of the mass m_1 is a circle of constant radius r_0 from the hole in the table? Find the radius of this circle in terms of l, m_1, m_2, and g. Let $E(r_0)$ be the total energy in this case. Prove that $E(r_0) = \frac{3}{2}m_2 g r_0$. Why is $E(r_0)$ the minimum possible total energy E?

P318 HW #4 SOLUTIONS DUE: 20 FEB

CORRECTIONS? pt267@cornell.edu

1. (H&F 4.10) <u>2 MASSES IN A GRAVITATIONAL FIELD</u>

Without loss of generality, assume that the constant gravitational field points in the $(-\hat{z})$-direction. The potential for the 2 point masses is

$$V_{total}(\vec{r}_1, \vec{r}_2) = \underbrace{m_1 g z_1 + m_2 g z_2}_{\text{GRAVITATIONAL POT.}} + \underbrace{V(\vec{r}_1 - \vec{r}_2)}_{\text{POTENTIAL WHEN } g \to 0}$$

As noted in the text (p.137), the gravitational potential can be written in terms of only the center-of-mass coordinate & total mass:

$$V_g = M g Z_{cm} \qquad M = m_1 + m_2$$
$$Z_{CM} = \frac{1}{M}(m_1 z_1 + m_2 z_2)$$

Note that the Lagrangian separates into two independent pieces:

$$L = \underbrace{\tfrac{1}{2} M \dot{\vec{R}}_{cm}^2 - M g Z_{cm}}_{L_{cm}} + \underbrace{\tfrac{1}{2}\mu \dot{\vec{r}}^2 - V(\vec{r})}_{L_{RELATIVE}}$$

THE CM MOTION CAN BE SOLVED FOR (TRIVIALLY!) & HAS NO EFFECT ON THE RELATIVE MOTION. BY THE ARGUMENTS of SECTION 4.4 IN H&F, THIS REDUCES TO A 1-BODY PROBLEM FOR $\vec{r}$ (BECAUSE THE Z_{cm} MOTION IS FREE FALL, INDEPENDENT OF $V(\vec{r})$).

2. (H & F 4.11) <u>N-body system</u>

We are asked to use Lagrangian methods, but note that this is trivial using Newtonian physics:
$\ddot{R}_{cm} = \frac{1}{\Sigma m_i} \Sigma m_i \ddot{\vec{r}} = \frac{1}{M} \sum_i \vec{F}_i$.

We want coordinates for the N-body system such that the center of mass $\vec{R}_{cm}$ is separated from the relative coordinates. The generalization of the 2-body coordinates is:

$$\vec{R}_{cm} = \left(\frac{1}{\Sigma m_i}\right) \sum_i m_i \vec{r}_i$$
$$\vec{R}_i = \vec{r}_i - \vec{r}_{i+1} \qquad i < N$$

In terms of $\vec{R}_{cm}$ & the $\vec{R}_i$'s, the original coordinates are of the form

$$\vec{r}_i = \vec{R}_{cm} + \cdots \leftarrow \text{INDEPENDENT OF } \vec{R}_{cm}$$

This should make sense intuitively, but we explicitly write $\vec{r}_i(\vec{R}_{cm}, \vec{R}_j)$ in the appendix.

In the 3N-dimensional configuration space, R_{cm} is perpendicular to the relative coordinates. Thus the kinetic term is $T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2} M \dot{\vec{R}}_{cm}^{\,2} + \cdots$, with no $R_{cm} R_i$ cross terms. SEE THE APPENDIX FOR AN EXPLICIT PROOF!

Assuming that the force $\vec{F}_i$ on the i^{th} particle is conservative, we may write

$$L = \sum_i \left[\tfrac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - V_i(\vec{r}_i) \right] \qquad \text{with } \vec{F}_i = -\underset{\partial/\partial\vec{r}_i}{\underset{\uparrow}{\vec{\nabla}_i}} V_i(\vec{r}_i)$$

In the $\vec{R}_{cm}, \vec{R}_i$ coordinates,

$$L = \tfrac{1}{2} M \dot{\vec{R}}_{cm}^{\,2} + \cdots - V_1(\vec{R}_{cm} + \cdots) - V_2(\vec{R}_{cm} + \cdots) - \cdots$$

Then the $\vec{R}_{cm}$ equation of motion is

$$M \ddot{\vec{R}}_{cm} + \underbrace{\vec{\nabla} V_1 + \vec{\nabla} V_2 + \cdots}_{-\sum_i \vec{F}_i} = 0$$

Thus the cm is acted upon by a force $\sum_i \vec{F}_i$.

3. (H&F 4.12) Exploding Projectile

a) Apply the result of problem 2.
Use CM coordinates:

$$\vec{R}_{cm} = \frac{1}{\Sigma m_i} \sum_i m_i \vec{r}_i$$
$$\vec{R}_{12} = \vec{r}_1 - \vec{r}_2$$
$$\vec{R}_{23} = \vec{r}_2 - \vec{r}_3$$

Then $V = V_A(\vec{R}_{12}) + V_B(\vec{R}_{23}) + V_c(\vec{R}_{12} + \vec{R}_{23})$
ie V is independent of $\vec{R}_{cm}$.
$\Rightarrow$ the R_{cm} equation of motion is

$$M\ddot{\vec{R}}_{cm} = 0 \Rightarrow \boxed{\dot{\vec{R}}_{cm} = \text{const.}} = V_{cm}$$

b) The total momentum is $\sum_i m_i \vec{V}_i = \sum_i m_i \dot{\vec{r}}_i$
$= M \vec{V}_{cm}$

IN THE CM FRAME, $\vec{V}_{cm} = 0$.
THUS: $m_1\vec{V}_1 + m_2\vec{V}_2 + m_3\vec{V}_3 = 0$.

$\Rightarrow$ $\vec{V}_1, \vec{V}_2, \vec{V}_3$ are linearly dependent & hence (IN THE CM FRAME) lie in a plane.

c) Now use the same analysis as problem 1. Choose coordinates so that the constant gravitational force is in the $(-\hat{z})$ direction.

The potential is modified by an additional set of terms:

$$\Delta V = \sum_i m_i g z_i$$

But this can be written in terms of $\vec{R}_{cm}$

$$\Delta V = M g z_{cm}$$

Thus the Lagrangian separates into two pieces that don't interact:

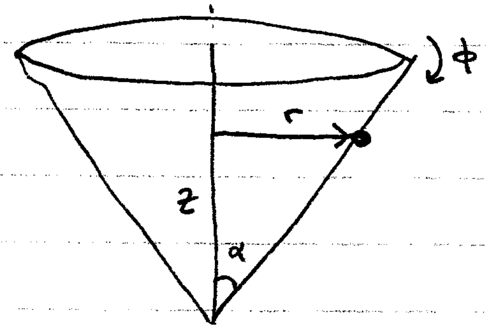
$$L = \underbrace{\tfrac{1}{2} M \dot{\vec{R}}_{cm}^{\,2} - M g z_{cm}}_{L_{cm}} + \underbrace{(R_{12}, R_{23} \text{ KIN. TERMS}) - V}_{L_{\text{RELATIVE}}}$$

$\boxed{\text{CENTER OF MASS FALLS LIKE A POINT MASS IN A CONSTANT GRAV. FIELD}}$

RELATIVE MOTION IS UNCHANGED.

CONTINUED

4. (H&F 4.13) Massive particle in a cone



$z = r\cot\alpha$

$V = mgr\cot\alpha$

$$\boxed{L = \frac{1}{2}m(r\dot{\phi})^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(\dot{r}\cot\alpha)^2 - mgr\cot\alpha}$$

observe: L independent of $\phi \Rightarrow l = mr^2\dot{\phi} = \text{const}$

EOM, r: $m\ddot{r} + m\cot\alpha\,\ddot{r} - mr\dot{\phi}^2 + mg\cot\alpha = 0$

$$\boxed{(1+\cot\alpha)\ddot{r} - \frac{l^2}{m^2r^3} + g\cot\alpha = 0}$$

$$H = \frac{1}{2}m(\dot{r}\cot\alpha)^2 + \underbrace{\frac{1}{2}m(r\dot{\phi})^2}_{\frac{1}{2}\frac{l^2}{mr^2}} + \frac{1}{2}m\dot{r}^2 + mgr\cot\alpha$$

$$\boxed{V_{eff}(r) = \frac{1}{2}\frac{l}{Mr^2} + mgr\cot\alpha}$$

$\boxed{\text{This separation required the force to be gravitational and constant.}}$

so that the force is proportional to mass

linear in $\vec{r}_i$

This allows the external potential to be written in terms of $\vec{R}_{cm}$ only.

$$V \sim \sum_i m_i \vec{r}_i \cdot \overrightarrow{\text{CONST}} \sim \vec{R}_{cm} \cdot \overrightarrow{\text{CONST}}$$

$\boxed{\text{The analysis in part b still holds in the noninertial cm frame, thus the fragments are still coplanar in this frame.}}$

FOR WHAT HORIZONTAL VELOCITY $(r\dot{\phi})$ WILL A PARTICLE AT $z_0 = r_0 \cot\alpha$ GIVE CIRCULAR MOTION?

PICK $\dot{\phi} \leftrightarrow l$ SUCH THAT r_0 IS THE MINIMUM OF V_{eff}. Physically: the particle doesn't fall despite gravity because this would cause it to rotate faster & hence increase its energy.
(THIS IS BECAUSE l IS CONSERVED!)

$$V'_{eff}(r_0) = -\frac{l^2}{mr_0^3} + mg\cot\alpha = 0$$

$$\Rightarrow l^2 = m^2 g r_0^3 \cot\alpha$$

$$\cancel{m^2}\underbrace{r_0^4 \dot{\phi}^2}_{r_0^2 (\underbrace{r_0\dot{\phi}}_{v_0})^2} = \cancel{m^2} g r_0^3 \cot\alpha$$

$$\Rightarrow \boxed{v_0^2 = g r_0 \cot\alpha = g z_0}$$

5. N=3 Polymer Model

$$L = \left(\sum_{i=1}^{3} \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}\right) - \lambda_1(\vec{r}_1 - \vec{r}_2)^2 - \lambda_2(\vec{r}_2 - \vec{r}_3)^2$$

EOM, $\vec{r}_1$: $\quad m\ddot{\vec{r}}_1 - 2\lambda_1(\vec{r}_1 - \vec{r}_2) = 0$

$\vec{r}_2$: $\quad m\ddot{\vec{r}}_2 + 2\lambda_1(\vec{r}_1 - \vec{r}_2) - 2\lambda_2(\vec{r}_2 - \vec{r}_3) = 0$

$\vec{r}_3$: $\quad m\ddot{\vec{r}}_3 + 2\lambda_2(\vec{r}_2 - \vec{r}_3) = 0$

CONSTRAINT EQUATIONS

$$(\vec{r}_1 - \vec{r}_2)^2 = d^2$$
$$(\vec{r}_2 - \vec{r}_3)^2 = d^2$$

take time derivatives: $\quad (i = 1, 2)$

$$(\vec{r}_i - \vec{r}_{i+1}) \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_{i+1}) = 0$$ (node spacing doesn't change)

$$|\dot{\vec{r}}_i - \dot{\vec{r}}_{i+1}|^2 + (\vec{r}_i - \vec{r}_{i+1}) \cdot (\ddot{\vec{r}}_i - \ddot{\vec{r}}_{i+1}) = 0$$

SIMPLIFY NOTATION: IN EOM, DIVIDE BY m

DEFINE $\tilde{\lambda} = \lambda / m$

$$\ddot{\vec{r}}_1 - 2\tilde{\lambda}_1(\vec{r}_1 - \vec{r}_2) = 0$$
$$\ddot{\vec{r}}_2 + 2\tilde{\lambda}_1(\vec{r}_1 - \vec{r}_2) - 2\tilde{\lambda}_2(\vec{r}_2 - \vec{r}_3) = 0$$
$$\ddot{\vec{r}}_3 + 2\tilde{\lambda}_2(\vec{r}_2 - \vec{r}_3) = 0$$

$$|\dot{\vec{r}}_1 - \dot{\vec{r}}_2|^2 + (\vec{r}_1 - \vec{r}_2) \cdot (\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2) = 0$$
$$|\dot{\vec{r}}_2 - \dot{\vec{r}}_3|^2 + (\vec{r}_2 - \vec{r}_3) \cdot (\ddot{\vec{r}}_2 - \ddot{\vec{r}}_3) = 0$$

Strategy massage linear combinations of EoM so that we can plug in the constraint eqs

We see that the constraint eqs include the terms $(\vec{r}_i - \vec{r}_{i+1}) \cdot (\ddot{\vec{r}}_i - \ddot{\vec{r}}_{i+1})$. Let's take dot products of the EoM to get similar terms:

$$\text{I} \quad \ddot{\vec{r}}_1 \cdot (\vec{r}_1 - \vec{r}_2) - 2\tilde{\lambda}_1 \underbrace{(\vec{r}_1 - \vec{r}_2)^2}_{= d^2} = 0$$

why squared? so that it's easy to use dimensional analysis to check our work

$$\text{II} \quad \ddot{\vec{r}}_3 \cdot (\vec{r}_2 - \vec{r}_3) + 2\tilde{\lambda}_2 d^2 = 0$$

$$\text{III} \quad \ddot{\vec{r}}_2 \cdot (\vec{r}_1 - \vec{r}_2) + 2\tilde{\lambda}_1 d^2 - 2\tilde{\lambda}_2 \underbrace{(\vec{r}_2 - \vec{r}_3)\cdot(\vec{r}_1 - \vec{r}_2)}_{\equiv p^2} = 0$$

$$\text{IV} \quad \ddot{\vec{r}}_2 \cdot (\vec{r}_2 - \vec{r}_3) + 2\tilde{\lambda}_1 p^2 - 2\tilde{\lambda}_2 d^2 = 0$$

Now take linear combinations of the desired form:

$$\text{I} - \text{III}: \quad (\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2)\cdot(\vec{r}_1 - \vec{r}_2) - 4\tilde{\lambda}_1 d^2 + 2\tilde{\lambda}_2 p^2 = 0$$

$$\text{IV} - \text{II}: \quad (\ddot{\vec{r}}_2 - \ddot{\vec{r}}_3)\cdot(\vec{r}_2 - \vec{r}_3) - 4\tilde{\lambda}_2 d^2 + 2\tilde{\lambda}_1 p^2 = 0$$

Define $a^2 = (\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2)\cdot(\vec{r}_1 - \vec{r}_2) = |\dot{\vec{r}}_1 - \dot{\vec{r}}_2|^2$

$b^2 = (\ddot{\vec{r}}_2 - \ddot{\vec{r}}_3)\cdot(\vec{r}_2 - \vec{r}_3) - |\dot{\vec{r}}_2 - \dot{\vec{r}}_3|^2$

thanks to Harry Cheng for this observation.

$$\text{V} = \text{I} - \text{III}: \quad a^2 = 4d^2\tilde{\lambda}_1 - 2p^2\tilde{\lambda}_2$$
$$\text{VI} = \text{IV} - \text{II}: \quad b^2 = 4d^2\tilde{\lambda}_2 - 2p^2\tilde{\lambda}_1$$

this is of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{pmatrix}$

WE KNOW THAT THE EIGENVECTORS ARE $\frac{1}{\sqrt{2}}(\tilde{\lambda}_1 \pm \tilde{\lambda}_2)$. THUS WE SOLVE BY TAKING $\text{V} \pm \text{VI}$:

$$a^2+b^2 = (4d^2-2p^2)\tilde{\lambda}_1 + (4d^2-2p^2)\tilde{\lambda}_2$$
$$= (4d^2-2p^2)(\tilde{\lambda}_1 + \tilde{\lambda}_2)$$

$$a^2-b^2 = (4d^2+2p^2)\tilde{\lambda}_1 - (4d+2p^2)\tilde{\lambda}_2$$
$$= (4d^2+2p^2)(\tilde{\lambda}_1 - \tilde{\lambda}_2)$$

Now it's easy to get $\tilde{\lambda}_1$ & $\tilde{\lambda}_2$

$$\tilde{\lambda}_1 = \frac{1}{2}\left(\frac{a^2+b^2}{4d^2-2p^2} + \frac{a^2-b^2}{4d^2+2p^2}\right)$$

$$\tilde{\lambda}_2 = \frac{1}{2}\left(\frac{a^2+b^2}{4d^2-2p^2} - \frac{a^2-b^2}{4d^2+2p^2}\right)$$

PLUGGING IN OUR DEFINITIONS

$$\boxed{\lambda_{1,2} = \frac{m}{2}\left(\frac{|\vec{r}_1-\vec{r}_2|^2+|\vec{r}_2-\vec{r}_3|^2}{4d^2-2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)} \pm \frac{|\vec{r}_1-\vec{r}_2|^2-|\vec{r}_2-\vec{r}_3|^2}{4d^2+2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)}\right)}$$

Revised EOM, w/o Lagrange multipliers:

$$\ddot{\vec{r}}_1 = \left(\frac{|\dot{\vec{r}}_1-\dot{\vec{r}}_2|^2+|\dot{\vec{r}}_2-\dot{\vec{r}}_3|^2}{4d^2-2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)} + \frac{|\dot{\vec{r}}_1-\dot{\vec{r}}_2|^2-|\dot{\vec{r}}_2-\dot{\vec{r}}_3|^2}{4d^2+2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)} \right)(\vec{r}_1-\vec{r}_2)$$

$$\ddot{\vec{r}}_2 = -\left(\frac{|\dot{\vec{r}}_1-\dot{\vec{r}}_2|^2+|\dot{\vec{r}}_2-\dot{\vec{r}}_3|^2}{4d^2-2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)} + \frac{|\dot{\vec{r}}_1-\dot{\vec{r}}_2|^2-|\dot{\vec{r}}_2-\dot{\vec{r}}_3|^2}{4d^2+2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)} \right)(\vec{r}_1-\vec{r}_2)$$
$$+ \left(\frac{|\dot{\vec{r}}_1-\dot{\vec{r}}_2|^2+|\dot{\vec{r}}_2-\dot{\vec{r}}_3|^2}{4d^2-2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)} - \frac{|\dot{\vec{r}}_1-\dot{\vec{r}}_2|^2-|\dot{\vec{r}}_2-\dot{\vec{r}}_3|^2}{4d^2+2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)} \right)(\vec{r}_2-\vec{r}_3)$$

$$\ddot{\vec{r}}_3 = -\left(\frac{|\dot{\vec{r}}_1-\dot{\vec{r}}_2|^2+|\dot{\vec{r}}_2-\dot{\vec{r}}_3|^2}{4d^2-2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)} - \frac{|\dot{\vec{r}}_1-\dot{\vec{r}}_2|^2-|\dot{\vec{r}}_2-\dot{\vec{r}}_3|^2}{4d^2+2(\vec{r}_1-\vec{r}_2)\cdot(\vec{r}_2-\vec{r}_3)} \right)(\vec{r}_2-\vec{r}_3)$$

Appendix: N-body coordinates

↳ THANKS TO SAM PARK!

GOAL: want a generalization of $\{\vec{R}_{cm}, \vec{r}\}$ for the N-body problem. Required properties:

I. SEPARATION INTO 1 CM COORDINATE $\vec{R}_{cm}$ AND (N-1) RELATIVE COORDINATES $\vec{R}_i$.

II. $\vec{r}_i = \vec{R}_{cm} + (\text{linear comb. of } \vec{R}_i)$

eg. important for problem #2

III $\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2 = \frac{1}{2} M \dot{\vec{R}}_{cm}^2 + \underline{(\text{no cross terms})} + (\vec{R}_i \text{ KIN. TERMS})$

Here's an explicit solution:

$$\vec{R}_{cm} = \frac{1}{M} \sum_i m_i \vec{r}_i \qquad \leftarrow M = \sum_i m_i$$

$$\vec{R}_i = \vec{r}_i - \vec{r}_{i+1} \qquad i < N$$

important remark: there are only (N-1) of these relative coordinates, but they span all N(N-1) possible relative distances. eg for $j > i$:

$$\vec{r}_i - \vec{r}_j = \sum_{k=i}^{j-1} \vec{R}_k$$

This satisfies I by construction.

To show II, we can be a little slick.
[Sam Park showed me how to do this]

Want to prove that $\exists$ some linear combination of the $\vec{R}_i$ such that $\vec{r}_i = \vec{R}_{cm} + \sum_j a_j \vec{R}_j$.

FOR SIMPLICITY, WRITE THIS AS $M\vec{r}_i = M\vec{R}_{cm} + \sum_j M a_j \vec{R}_j$

<u>OBSERVE</u>: $M\vec{R}_{cm}$ CONTAINS $m_i \vec{r}_i + \sum_{j \neq i} m_j r_j$

WANT TO ADD TERMS S.T. COEFF IS Σm_i

WANT TO KILL THESE TERMS

TO GET THE RIGHT $\vec{r}_i$ PREFACTOR: we have to add
$\sum_{j \neq i} m_j \vec{r}_i$

TO KILL THE UNWANTED TERMS, ADD
$-\sum_{j \neq i} m_j \vec{r}_j$

combine these:

$$\sum_j M a_j \vec{R}_j \stackrel{?}{=} \sum_{j \neq i} m_j (\vec{r}_i - \vec{r}_j)$$

is this possible? $\rightarrow$ <u>YES</u>: WE JUST SAID THAT THE $\vec{R}_i$ INDEED SPAN ALL POSSIBLE RELATIVE POSITIONS $(\vec{r}_i - \vec{r}_j)$

So II is satisfied.

What about ~~#~~, cross terms?

$$\dot{\vec{r}}_i = \dot{\vec{R}}_{cm} + \frac{1}{M}\sum_{j\neq i} m_j(\dot{\vec{r}}_i - \dot{\vec{r}}_j)$$

Want to prove that $\forall\, i,j$, the sum $\sum_k m_k \dot{\vec{r}}_k^{\,2}$ contains no cross terms $\dot{\vec{R}}_{cm}(\dot{\vec{r}}_i - \dot{\vec{r}}_j)$.

such terms appear when $k = i, j$:

$$\sum_k m_k \dot{\vec{r}}_k^{\,2} = \cdots + m_j \dot{\vec{r}}_j^{\,2} + \cdots + m_i \dot{\vec{r}}_i^{\,2} + \cdots$$

$$= m_j\left(\frac{1}{M} m_i (\dot{\vec{r}}_i - \dot{\vec{r}}_j) + \cdots\right)\dot{\vec{R}}_{cm} + m_i\left(\frac{1}{M} m_j(\dot{\vec{r}}_j - \dot{\vec{r}}_i) + \cdots\right)\dot{\vec{R}}_{cm} + \cdots$$

THESE HAVE EQUAL MAGNITUDE BUT OPPOSITE SIGN, SO THEY CANCEL.

$\Rightarrow \forall\, i,j$, there is no $\dot{\vec{R}}_{cm}\cdot(\dot{\vec{r}}_i - \dot{\vec{r}}_j)$ cross term!

Thus we are satisfied that nice N-body coordinates exist.

We can also be more explicit & write $\vec{r}_i$ in terms of our N coordinate vectors $\{\vec{R}_{cm}, \vec{R}_i\}$.

Want to show II, need algorithm to construct the particular linear combination of $\vec{R}_i$ that gives $M\vec{r}_i = M\vec{R}_{cm} + (\sum_j a_j \vec{R}_j)$

i) START W/ R_{cm}. THIS GIVES A 'GOOD TERM' $m_i \vec{r}_i$ PLUS 'JUNK TERMS' $\sum_{k\neq i} m_k \vec{r}_k$.

ii) STARTING @ $\vec{R}_1$: ADD A MULTIPLE OF $\vec{R}_1$ SUCH THAT THE $m_1 \vec{r}_1$ TERM IS CANCELED. THIS GENERATES AN $+m_1 \vec{r}_2$ JUNK TERM. SO THEN ADD A MULTIPLE OF $\vec{R}_2$ TO CANCEL BOTH THE $m_2 \vec{r}_2$ IN R_{cm}, BUT THE $m_1 \vec{r}_2$ AS WELL.

iii) CONTINUE DOING THIS — ADDING $(\sum_{k\leq j} m_k) R_j$ — UNTIL YOU REACH $\vec{R}_i$. FOR $\vec{R}_i$, ADD $(\sum_{k\neq i} m_i) \vec{R}_i$ SO THAT THE COEFFICIENT OF $\vec{r}_i$ IS M.

iv) NEXT REMOVE THE JUNK TERMS for $\vec{r}_j$ w/ $j > i$. (note that the $m_j \vec{r}_j$ in $\vec{R}_{cm}$ automatically cancels against 'junk' from the $\vec{R}_{j-1}$ correction).

vi) CONTINUE UNTIL THE END, ADDING TERMS $(\sum_{k>j} m_k) R_j$. OBSERVE THAT THE $j = N-1$ TERM WILL TERMINATE THE CORRECTIONS & WILL CANCEL THE LAST $m_N \vec{r}_N$ TERM.

THE EXPLICIT FORMULA IS

$$\boxed{\begin{aligned} M\vec{r}_i = M\vec{R}_{cm} &- \sum_{j<i} \Big(\sum_{k\leq j} m_k\Big) R_j \\ &+ \Big(\sum_{k\neq i} m_k\Big) R_i \\ &+ \sum_{j>i} \Big(\sum_{k>j} m_k\Big) R_j \end{aligned}}$$

FOR EXAMPLE: TO GET $\vec{r}_3$ IN $N=5$; HERE'S A TABLE OF COEFFICIENTS FOR THE VARIOUS $\vec{r}_i$ COMING FROM THE $\vec{R}$ TERMS:

	R_{cm}	R_1	R_2	R_3	R_4
r_1:	m_1	$-m_1$			
r_2:	m_2	$+m_1$	$-m_1-m_2$		
r_3:			$+m_1+m_2$	$+m_4+m_5$	
r_4:				$-m_4-m_5$	$+m_5$
r_5:					$-m_5$

observe: all coefficients of the $\vec{r}_i$ vanish except for $\vec{r}_3$, which has coefficient M, as desired.

Next check that there are no cross terms. ie $\forall a$, $\sum m_i \dot{\vec{r}}_i^{\,2}$ does not contain $R_{cm} R_a$.

To show this, go through each term in $\frac{1}{2}\sum m_i \dot{\vec{r}}_i^{\,2}$ and identify the $R_{cm} R_a$ coefficient:

$$\sum_{b>a} m_b \left(\sum_{k\le a} - m_k\right) \qquad \leftarrow \text{from all } \tfrac{1}{2} m_b \dot{\vec{r}}_b^{\,2} \text{ w/ } b>a$$

$$+ \quad m_a \left(\sum_{k\ne a} m_k\right) \qquad \leftarrow \text{from } \tfrac{1}{2} m_a \dot{\vec{r}}_a^{\,2} \text{ term}$$

$$+ \sum_{c<a} m_c \left(\sum_{k>a} m_k\right) \qquad \leftarrow \text{from } \tfrac{1}{2} m_c \dot{\vec{r}}_c^{\,2} \text{ w/ } c<a$$

the last 2 terms combine: $\sum_{c\le a} \sum_{k>a} m_c m_k$

first term: $\sum_{b>a} \sum_{k\le a} \left(-m_b m_k\right)$

since sums commute & dummy indices don't matter, these indeed cancel!