Sub Lecture: 2 body problem

vs. "The" 2 body problem

TODAY: 2 BODY STUFF — no homework!
FRI: Mo' LECTURES, REVIEW SESSION
MON: Prelim!

2-body problem:

2 point masses, $m_1 @ \vec{r}_1$ & $m_2 @ \vec{r}_2$ subject to a conservative force that depends only on $(\vec{r}_1 - \vec{r}_2)$

$\Rightarrow$ completely solvable in Lagrangian formalism

$$L = \tfrac{1}{2} m_1 \dot{\vec{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\vec{r}}_2^2 - \underline{V(\vec{r}_1 - \vec{r}_2)}$$

GENERAL FOR NOW. WILL FOCUS LATER ON GRAV & ELECTROSTATIC

cm motion is trivial in isolated sys (no ext f.) so use cm as a coord:

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

327 STUDENTS: THIS IS THE MONOPOLE MOMENT!

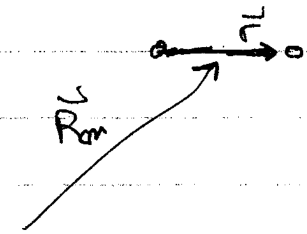
since $V = V(\underbrace{\vec r_1 - \vec r_2})$

use $\vec r = \vec r_1 - \vec r_2$ as second coord.
[not quite a dipole moment]
BUT for 2 BODY, $(\vec R, \vec r)$ A COMPLETE BASIS.

<u>You found</u>:

$$\vec r_1 = \vec R + \frac{M_2}{M_1+M_2}\vec r$$

$$\vec r_2 = R - \frac{M_1}{M_1+M_2}\vec r$$



Plug into kinetic energy:

$$T = \tfrac{1}{2} m_1 \left|\dot{\vec R} + \frac{m_2}{m_1+m_2}\dot{\vec r}\right|^2 + \tfrac{1}{2} M_2 \left|\dot{\vec R} - \frac{M_1}{M_1+M_2}\dot{\vec r}\right|^2$$

$$= \tfrac{1}{2}\underbrace{(M_1+M_2)}_{\text{TOTAL MASS}} |\dot{\vec R}|^2 + \tfrac{1}{2}\underbrace{\frac{M_1M_2^2 + M_2M_1^2}{(M_1+M_2)^2}} |\dot{\vec r}|^2$$

$$\left(\frac{M_1M_2}{M_1+M_2}\right) \equiv \mu$$

REDUCED MASS

no cross term!

cf "total charge"

$$L = \frac{1}{2}(M_1+M_2)|\dot{\vec{R}}|^2 + \frac{1}{2}\mu|\dot{\vec{r}}|^2 - V(\vec{r})$$

indep of $\vec{R}$

$$\Rightarrow \vec{P}_R = \frac{\partial L}{\partial \dot{\vec{R}}} = \underbrace{(M_1+M_2)\dot{\vec{R}}}_{\text{TOTAL MOMENTUM}} \quad \text{CONSERVED}$$

is it obvious? yes, eg intermolecular forces which keeps things together as I throw them in space.

NO EXTERNAL FORCES $\leftrightarrow$ no $V(\vec{R})$

RELATIVE POSITION

$$\frac{\partial L}{\partial \vec{r}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}}\right)$$

$$\boxed{-V'(|\vec{r}|)\underbrace{\frac{\partial|\vec{r}|}{\partial \vec{r}}}_{\hat{r}} = \mu\ddot{\vec{r}}} \Rightarrow \mu\ddot{\vec{r}} = -V'(|\vec{r}|)\,\hat{r}$$

in the $\hat{r}$ DIRECTION.

$V'(\cdots)$ derivative w/r/t arg.

& we used: V is scalar w/ vec. arg, can only depend on $\vec{r}^2 = |\vec{r}|^2$

one body problem

for point mass (μ) subj. to central force $\vec{F} = -V'(|\vec{r}|)\,\hat{r}$

$\vec{r}$ & $\vec{v} = \dot{\vec{r}}$ define a plane w/ normal vec $\vec{v} \times \vec{r}$.

~~note~~: $\frac{d}{dt}(\dot{\vec{r}} \times \vec{r}) = \ddot{\vec{r}} \times \vec{r} + \cancel{\dot{\vec{r}} \times \dot{\vec{r}}} = 0$

0

BUT $\vec{F} \sim \hat{r}$ FROM BEFORE

thus normal vec to plane is constant
$\Rightarrow$ plane is constant.

W/o log: let $\vec{r} \in$ xy plane

ignore from now on: $L = L_{cm} + L_{rel}$, SEPARABLE

$L = [\text{KE of cm}] + \frac{1}{2}\mu|\dot{\vec{r}}|^2 - V(|\vec{r}|)$

Go to polar: $x = r\cos\theta$
$y = r\sin\theta$

$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

indep of θ: $L_z = l = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{const}$.

angular momentum

so: original problem: 2×3 DOF
now: 1 dof, dep on V:

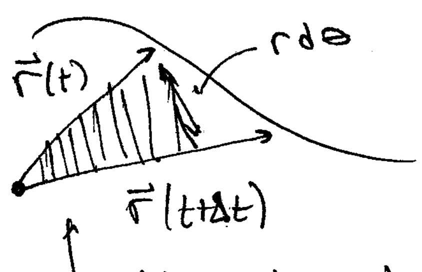
$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$\Rightarrow \underbrace{\mu r \dot{\theta}^2}_{\frac{l^2}{\mu r^3}} - V'(r) = \mu \ddot{r}$$

$$\Rightarrow \boxed{\mu \ddot{r} = \frac{l^2}{\mu r^3} - V'(r)}$$

we will study this in detail
$\rightarrow$ elliptical shape of orbits, etc.

Kepler : $l =$ const



$$dA = \tfrac{1}{2} r^2 \, d\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \underbrace{\frac{l}{2\mu}}_{\text{const}} \ldots$$

did this dep on GRAV? inv. square?

conserved quantities like l are called
FIRST INTEGRALS (b/c EOM ARE 2^{ND} O DIFF EQ
& CONS QUANTITIES ARE 1^{st} O)

the other obvious one is the Hamiltonian / E

$$E = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) \quad = 2T - L$$

$$= \tfrac{1}{2}\mu\dot{r}^2 + \underbrace{\tfrac{1}{2}\frac{l^2}{\mu r^2} + V(r)}_{V_{eff}}$$

$\Uparrow$ cuidado

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E-V) - \frac{l^2}{\mu^2 r^2}}$$

more gen: $E = \tfrac{1}{2}\dot{q}^2 + V \rightarrow \dot{q} = \sqrt{2(E - V(q))}$

$$dt = \frac{dq}{\pm\sqrt{2(E-V(q))}}$$

instead, can also do

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}\,dr = \dot{\theta}/\dot{r}\,dr \quad w/ \quad \dot{\theta} = l/\mu r^2$$

$$\theta(r) = \int \frac{\pm(l/r^2)\,dr}{\sqrt{2\mu\left(E - V - l^2/2\mu r^2\right)}}$$

$$M R_{cm} = m_i r_i$$
$$R_i = r_i - r_{i+1}$$

$$M r_i = M R_{cm} - \sum_{j<i} \left(\sum_{k\le j} m_k \right) R_j$$
$$+ \left(\sum_{k\neq i} m_k \right) R_i$$
$$+ \sum_{j>i}^{N-1} \left(\sum_{k=j+1}^{N-1} m_k \right) R_j$$

no x terms.

eg:

	R_{cm}	R_1	R_2	R_3	R_4
r_1	m_1	$-m_1$			
r_2	m_2	$+m_1$	$-m_1 - m_2$		
r_3	m_3		$+m_1 + m_2$	$+m_4 + m_5$	
r_4	m_4			$-m_4 - m_5$	$+m_5$
r_5	m_5				$-m_5$

$$\sum_{b>a} m_b \left(\sum_{k\le a} - m_k \right)$$

$$m_a \left(\sum_{k\neq a} m_k \right)$$

$$\sum_{c<a} m_c \left(\sum_{k>a} m_k \right)$$

$$\sum_{c\le a} m_c \sum_{k>a} m_k$$