P318 Lec

6 March

- CONTINUOUS SYMMETRIES OF L
- CONSERVATION LAWS FOR FREE

HW #2

We've said many times:

translation sym → momentum conservation
"homogeneity of space"

time translation sym → E (H) conservation
"homogeneity of time"

space-time related by relativity
⇒ ENERGY & MOMENTUM AS WELL

How do we make this precise?

L ← encodes our theory

↑

symmetries of this guy → symmetries of theory

1st invariance

Noether's thm: symmetries of Lagrangian w/rt continuous transformations → CONSERVED QUANTITIES.

↳ gives mathematical procedure

LOOK @ L, OBSERVE SYMMETRY

PLUG INTO MACHINERY → CONSERVED QUANTITY.

You've already seen this!

eg: $L(q,\dot{q})$ indep of q → $\partial L/\partial \dot{q}$ conserved!

→ special simple case when symmetry is "obvious"

eg $q \to q + a$. OFTEN THIS SIMPLENESS IS HIDDEN.

So what? often the intuitive / geometric ~~interpretation~~ understanding of a symmetry is elusive

↳ but the PUNCHLINE is evident

- conservation of charge
  baryon #
  etc.

eg: Free particle

Lie group

$\Downarrow$ eg UNITARY TRANSF IN QM

CONTINUOUS

$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2)$$

this Lagrangian possesses a mysterious continuous symmetry:
TRANSFORMATION OF SYSTEM DESCRIBED BY A CONTINUOUS PARAMETER s ← let's call it "s-sym"

$s = 0$ no transformation

since s is continuous, consider $\underbrace{\text{small transformations}}$

Lie algebra
or generators
(in QM: OPERATORS THAT ENACT YOUR TRANSF.)
eg HERMITIAN

$$r \to r + \underbrace{\cos\theta\,\delta s}_{\delta r} \qquad \theta \to \theta - \underbrace{\frac{\sin\theta}{r}\,\delta s}_{\delta\theta}$$

Verify that this is a symmetry

$$\dot r \to \dot r + \delta\dot r = \dot r - (\sin\theta)\dot\theta\,\delta s$$

$$\dot\theta \to \dot\theta + \delta\dot\theta = \dot\theta - \frac{\cos\theta}{r}\dot\theta\,\delta s + \frac{\sin\theta}{r^2}\dot r\,\delta s$$

$\swarrow$ finite transf = exponentiation of infinitesimal

infinitesimal δs, keep only first order

$$\dot{r}^2 + r^2\dot\theta^2 \Rightarrow (\dot r + \delta\dot r)^2 + (r+\delta r)^2(\dot\theta + \delta\dot\theta)^2$$

$$= \dot r^2 + r^2\dot\theta^2$$

$$+ \underbrace{2\dot r\delta\dot r + 2r\delta r\dot\theta^2 + r^2 \cdot 2\dot\theta\delta\dot\theta}_{\mathcal{O}(\delta s)} + \mathcal{O}(\delta s^2)$$

$$= \dot r^2 + r^2\dot\theta^2$$

cancel!
$$\left\{ \begin{aligned} &+ 2\dot r(-\sin\theta\,\dot\theta\,\delta s) \\ &+ 2r(\cos\theta\,\delta s)\dot\theta^2 \\ &+ 2r^2\dot\theta\left(-\tfrac{1}{r}\cos\theta\,\dot\theta + \tfrac{\dot r}{r^2}\sin\theta\right)\delta s \end{aligned}\right. + \mathcal{O}(\delta s^2)$$

invariance of L is exact in lim of small δs.

Let's formalize & GENERALIZE this.

START w/ $L(q_1, q_2, \cdots, \dot q_1, \dot q_2, \cdots)$

$q_1 \longmapsto Q_1(s) \leftarrow$ continuous parameter

$q_2 \longmapsto Q_2(s)$ s.t. $Q_i(0) = q_i$

$\vdots$

"symmetry" (INVARIANCE) $\mapsto \left.\dfrac{d}{ds} L\left(Q_1(s), \cdots, \dot Q_1(s), \cdots\right)\right|_{s=0} = 0$

this is why a continuous sym was nec.

note: $Q_1 = Q_1(s,t)$

$\cdot = d/dt$

$$= \sum_{i=1}^{N} \left(\frac{\partial L}{\partial Q_i} \frac{dQ_i}{ds} + \frac{\partial L}{\partial \dot{Q}_i} \frac{d\dot{Q}_i}{dS} \right) \Bigg|_{S=0}$$

$\Uparrow$

$$= \frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_i} \leftarrow \text{EOM. } (\forall \text{ fixed } S)$$

remark: on-shell condition.

[Q: what happens in QM?]

↳ WARD-TAKAHASHI; GENERATORS

$$= \Sigma \left[\underbrace{\frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_i} \frac{dQ_i}{dS} + \frac{\partial L}{\partial \dot{Q}_i} \frac{d}{dt} \frac{dQ_i}{dS}}_{= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i} \frac{dQ_i}{dS}\right) \text{ by chain rule}} \right] \Bigg|_{S=0}$$

next recall: $\displaystyle \frac{\partial L}{\partial \dot{Q}_i}\Big|_{S=0} = \frac{\partial L}{\partial \dot{q}_i} = \boxed{P_i}$

$$= \frac{d}{dt} \Big(\underbrace{\Sigma\, P_i \frac{dQ_i}{dS} \Big|_{S=0}}_{\equiv I,\ \text{conserved}} \Big) = 0$$

So let's identify I for our mysterious symmetry acting on the free particle:

$$P_r = m\dot{r} \qquad dQ_r/ds\big|_{s=0} = \cos\theta$$

$$P_\theta = mr^2\dot{\theta} \qquad dQ_\theta/ds\big|_{s=0} = -(\sin\theta)/r$$

$$I = m\dot{r}\cos\theta - mr^2\dot{\theta}\,\frac{\sin\theta}{r}$$

$$= m(\underbrace{\dot{r}\cos\theta - r\sin\theta}\,\dot{\theta})$$

Q: What is this?

$$x = r\cos\theta$$
$$\dot{x} = \dot{r}\cos\theta - r\sin\dot{\theta}$$

$$\Rightarrow I = m\dot{x} = p_x$$

So "s-symmetry" was just TRANSLATION sym!
$x \longrightarrow x+s$.

↳ "obvious" in cartesian ... "L indep of x"
BUT HARD TO SEE IN RADIAL!

similarly: angular momentum cons is harder to see in cartesian rather than polar.

Noether gives way to write it.

extra

Some questions

but we used on-shellness!

1. What happens in QM? (OFF-SHELL)

answer: invariance of L still an invariance of the PATH INTEGRAL (SUM OVER HISTORIES)

TURNS OUT: conserved quantity given by Noether procedure → GENERATOR of the symmetry (OPERATOR). MEANING

$Q \mapsto \hat{Q}$, HERMITIAN s.t.
$U_Q = e^{i\hat{Q}}$ ENACTS THE SYMMETRY

Remark: IN field theory (NOT NEC QUANTUM), Noether's thm EVEN MORE POWERFUL
→ GIVES CONSERVED CURRENT ← conservation in space vs in time.
eg. ELECTROMAGNETIC CURRENT

2. What about conservation of charge?

$\psi \to e^{i\theta}\psi$

comes from phase invariance of QM! recast in terms of L:

$\underbrace{\Sigma e^{iS/\hbar}}_{\mathbb{C}\ \#\text{ of unit length}}$ ⇗ $S \in \mathbb{R}$. but $\psi(x)$ is $\mathbb{C}$.
→ $\cancel{\$}\, L(\psi) = L(\psi^*\psi)$

leads to 3rd Q: why did we care about invariance of L ... seems like the more general ~~concept~~ requirement is invariance of S!

in addition to $q \to Q(s)$ ← for simplicity, just 1 coord.

$t \to T(s)$

$Q = q + \delta q$

$T = t + \delta t$

INFINITESIMAL $\delta q, \delta t \sim \delta s$

NOT VARIATIONAL CALCULUS!!

$$S = \int_{t_0}^{t_1} dt\, L \mapsto \int_{t_0+\delta t_0}^{t_1+\delta t_1} \left(L(q,\dot q,t) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q \right)$$

$$= \left(\int_{t_0}^{t_1} dt\, L\right) + L(\cdots,t_1)\delta t_1 - L(\cdots,t_0)\delta t_0$$

$$+ \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\delta q(t)$$

from integration by parts

$$+ \frac{\partial L}{\partial \dot q}(\cdots,t_1)\,\delta q(t_1)$$

$$- \frac{\partial L}{\partial \dot q}(\cdots,t_0)\,\delta q(t_0)$$

note: NOT PATH VARIATION (fix END PTS)

→ VARIATION OF SYSTEM!

$$0 = \int_{t_0}^{t_1} \frac{d}{dt}\Big[\underbrace{\left(L - \frac{\partial L}{\partial \dot q}\dot q\right)}_{= -H} \delta t + \underbrace{\frac{\partial L}{\partial \dot q}\delta q}_{I \text{ from earlier!}} \Big]$$

Hamiltonian has arrived!

4. is there a way to make hard-to-see symmetries easier to work with?

eg. $\partial L/\partial q = 0 \rightarrow \partial L/\partial \dot{q} = \text{CONST.}$

↳ yes! [raison d'être for Hamiltonian formulation!]

5. we've seen: symm → cons. ~~quant~~ quantity

a) ALL symm → cons quantity?

↓ COUNTER EX

- DISCRETE SYM → yes, but not by NOETHER PROCEDURE
- "DYNAMICAL SYM" → not sym of L!
  ↳ not in L, but in EOM
  eg. $\dot{q} \rightarrow \dot{q} + v$ (GALILEAN)
  or $q \rightarrow \lambda q$

related to cons. of charge! [but not really--]

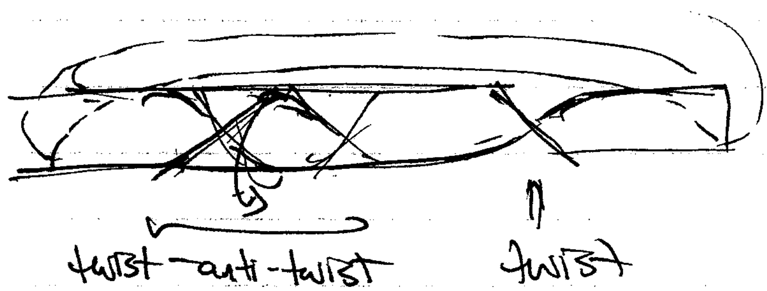
- GAUGE SYM → redundancy of ~~desc~~ DESCRIPTION
  ↳ "local symmetries" (eg $s = s(t)$)
  VERY POWERFUL/DEEP/ETC ... Noether's 2^{ND} THM.
  → think about E&M lagrangian
  $F_{\mu\nu}F^{\mu\nu}$ inv. under $A \rightarrow A + \partial\lambda$

b) ALL CONSERVED QUANTITIES
from Noether procedure?

↳ almost.

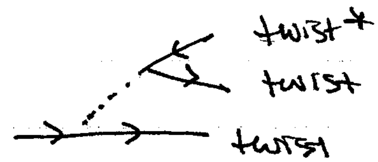
main exception: topological quantities.

analogs of mobius strips



FOR EXAMPLE
tetherball
winding!

↳ by the way:



↳ "solitons"
(only in field theories)

6) What about inequivalent L?

↳ we saw trivial examples

$L \to aL, \quad L \to L + a$

<u>NONTRIVIAL</u> : $L_1 = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \omega^2(q_1^2 + q_2^2)$

$$L_2 = \underbrace{\dot{q}_1 \dot{q}_2}_{\text{quad in velocity}} - \omega^2 q_1 q_2$$

⟶ same EOM

L_1 : $q_1 \to Q_1 = \cos\alpha \, q_1 + \sin\alpha \, q_2$
$q_2 \to Q_2 = -\sin\alpha \, q_1 + \cos\alpha \, q_2$

L_2 : $q_1 \to Q_1 = s q_1$
$q_2 \to Q_2 = \frac{1}{s} q_2$

<u>next time</u> : Hamiltonians