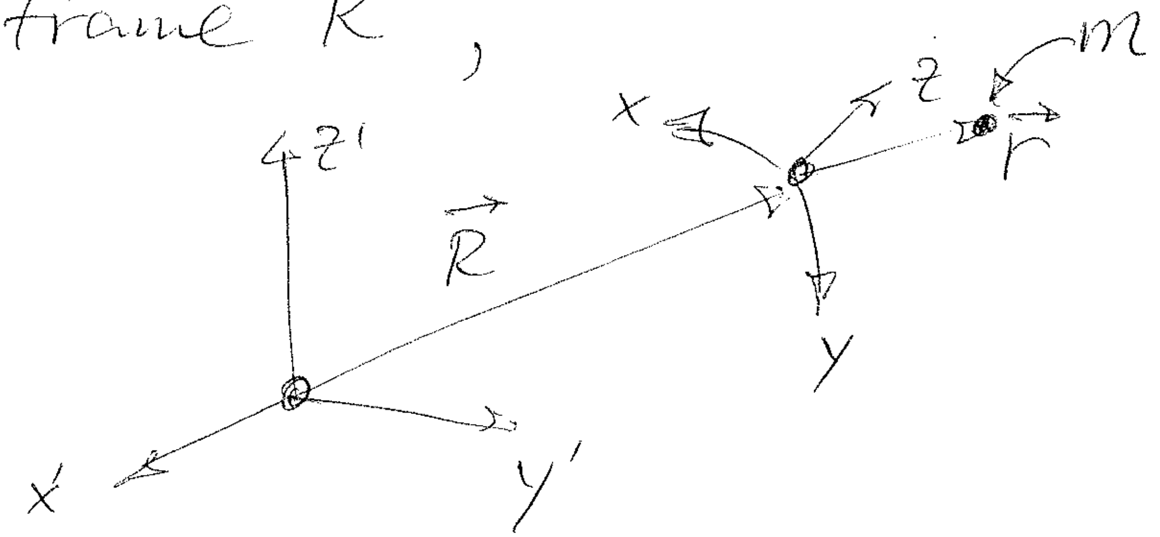
Rigid-body dynamics

- Kinetic energy

After the kinematic exercises of the previous lectures concerning rotating frames, ~~we~~ where the angular velocity $\vec{\omega}$ was under the control of an external agent, we now turn to actual dynamics. We will find that even the free motion of a rigid body, that is when there are no external forces & torques, can be surprisingly rich. As an example, the body frame associated with a football (NFL not FIFA) thrown into space, in general has a time-dependent $\vec{\omega}$. One of our main goals will be to obtain the equations of motion for $\vec{\omega}(t)$.

Let's say we have a <u>rigid body</u>, such as a ~~foot~~ football, where each point $\vec{r}$ of the body is fixed in some body frame K (so $\overset{\circ}{\vec{r}} = 0$). If we also allow the origin of the body frame to be translated by $\vec{R}(t)$ relative to our inertial space frame K',



then the velocity $\vec{V}$ of the point $\vec{r}$ is

$$\vec{V} = \mathring{\vec{R}} + \vec{\omega} \times \vec{r}.$$

If mass m is concentrated at point $\vec{r}$ of the rigid body, its contribution to the kinetic energy will be,

$$\frac{1}{2}m|\vec{v}|^2 = \frac{1}{2}m|\dot{\vec{R}}|^2 + m\dot{\vec{R}}\cdot\vec{\omega}\times\vec{r} + \frac{1}{2}m(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})$$

We can model a general rigid body as a collection of points $\vec{r}_i$ with associated masses m_i. The total kinetic energy will then be the expression above, summed over i. It's convenient to choose the origin of the rigid body to be its center of mass, because then

$$\sum_i m_i \vec{r}_i = 0$$

and the second term in

the kinetic energy sum vanishes. In that case,

$$T = \underbrace{\tfrac{1}{2} M |\dot{\vec{R}}|^2}_{T_{trans}} + \underbrace{\tfrac{1}{2} \sum_i m_i |\vec{\omega} \times \vec{r}_i|^2}_{T_{rot}}$$

Here $M = \sum_i m_i$ is just the total mass. Since the rotational dynamics is only determined by T_{rot}, we will trurn our attention to this part of the kinetic energy.

First we use some vector identities:

$$(\vec{\omega} \times \vec{r}) \cdot \vec{\omega} \times \vec{r} = \vec{\omega} \cdot \vec{r} \times (\vec{\omega} \times \vec{r})$$

$$= \vec{\omega}\cdot\left(r^2\vec{\omega}-(\vec{\omega}\cdot\vec{r})\vec{r}\right)$$

Switching to index notation, with the Einstein sum. convention, i.e. $\vec{\omega}\cdot\vec{r} = \omega_\alpha r_\alpha$, the above expression becomes

$$r^2\omega_\alpha\omega_\alpha - r_\alpha r_\beta\,\omega_\alpha\omega_\beta$$

$$= \underbrace{(r^2\delta_{\alpha\beta}-r_\alpha r_\beta)}_{\text{depends on } i}\ \underbrace{\omega_\alpha\,\omega_\beta}_{i\text{-independent}}$$

Finally, performing the sum on i we obtain

$$T_{rot} = \frac{1}{2}\,\omega_\alpha I_{\alpha\beta}\,\omega_\beta$$

where

$$I_{\alpha\beta} = \sum_i m_i (r_i^2 \delta_{\alpha\beta} - r_{\alpha i} r_{\beta i})$$

defines the symmetric moment-of-inertia tensor. We have to choose a consistent basis of vectors to define the components ω_α and r_α. The best choice, of course, is the body basis because then the components r_α are constant and therefore so is $I_{\alpha\beta}$. As a result, the components ω_α also refer to the body frame.

We can also write the rotational kinetic energy using matrix notation:

$$T_{rot} = \frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_z\end{pmatrix}\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & \cdots & \\ & & \end{pmatrix}\begin{pmatrix}\omega_x \\ \omega_y \\ \omega_z\end{pmatrix}$$

$$= \frac{1}{2}\,\omega^T I\, \omega \,.$$

Using a constant orthogonal matrix U we can rotate the axes of the body frame to a new, possibly more convenient set of axes like this:

$$\omega \longrightarrow U\omega\,, \quad I \longrightarrow U I U^T$$

Since I is symmetric, we know

that we can choose U so the transformed matrix UIU^T is diagonal:

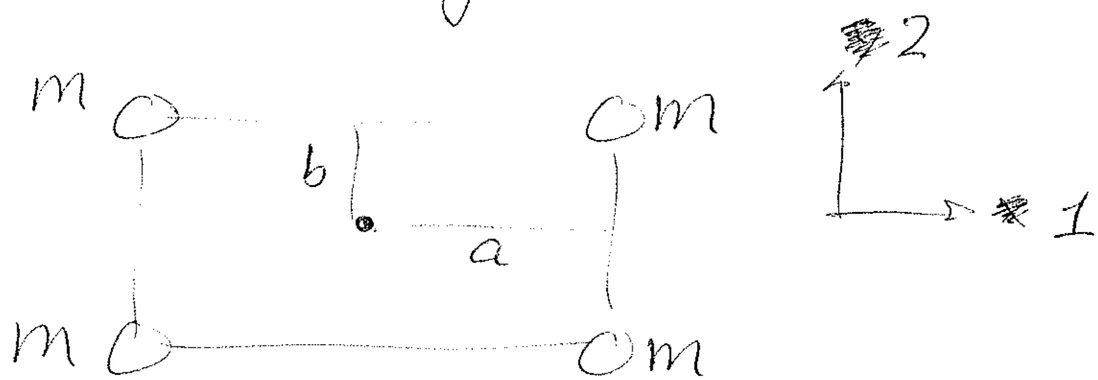
$$U\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad UIU^T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

The diagonal elements of the transformed moment of inertia matrix are called the "principal moments".
The principal moments must all be non-negative because

$$T_{rot} = \tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} I_2 \omega_2^2 + \tfrac{1}{2} I_3 \omega_3^2$$

must be non-negative (as a kinetic energy) for all three special cases of just one non-zero ω component.

The choice of axes within the body (the matrix U) that makes $\underline{I}$ diagonal can often be inferred from symmetry. As an example, consider four equal masses arranged in a rectangle:



Choosing axes along the rectangle edges, we find

$$I_{11} = 4m(a^2+b^2 - a^2)$$
$$I_{22} = 4m(a^2+b^2 - b^2)$$
$$I_{33} = 4m(a^2+b^2 - 0)$$

and all off-diagonal elements zero.

$$\underline{I} = \begin{pmatrix} 4mb^2 & 0 & 0 \\ 0 & 4ma^2 & 0 \\ 0 & 0 & 4m(a^2+b^2) \end{pmatrix}$$

The dynamical principle we will use to find the <u>torque-free</u> motion of a rigid body is the <u>constancy</u> of its <u>angular momentum</u>. We therefore need to express $\vec{L}$ in terms of $\vec{\omega}$. Consider, as we did earlier, our rigid body as a collection of masses m_i located at $\vec{R}+\vec{r}_i$, and having velocity $\dot{\vec{R}} + \vec{\omega}\times\vec{r}_i$. From the definition of angular ~~velocity~~ momentum we find

$$\vec{L} = \sum_i m_i (\vec{R} + \vec{r}_i) \times (\dot{\vec{R}} + \vec{\omega} \times \vec{r}_i)$$

$$= \underbrace{\vec{R} \times (M\dot{\vec{R}})}_{\vec{L}_{CM}} + \underbrace{\sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)}_{\vec{L}_{rot}}$$

As before, $M = \sum m_i$ is the total mass and some of the terms have vanished by choosing the body-origin to be the center of mass. $\vec{L}_{CM}$ is the angular momentum associated with the total mass moving with respect to our space frame. We will focus on $\vec{L}_{rot}$, the angular momentum the body may have even when its center of mass is at rest in the space frame. Using vector identities as in the kinetic energy calculation, we obtain

$$\vec{L}_{rot} = \sum_i m_i \left(r_i^2 \vec{\omega} - \vec{r}_i (\vec{r}_i \cdot \vec{\omega}) \right)$$

$$L_\alpha = \sum_i m_i \left(r_i^2 \delta_{\alpha\beta} - r_{\alpha i} r_{\beta i} \right) \omega_\beta$$

$$= I_{\alpha\beta} \omega_\beta .$$