Dynamics in phase space (cont.)

- canonical transformations
- Poisson bracket
- generating functions

For simplicity we'll consider systems with 1 DoF for which there are just two phase space variables: p and q. The generalization to multiple DoF will be given after we've seen all the phase-space concepts intoduced with just 1 DoF.

Consider the most general, time independent, transformation of phase-space variables:

$$Q = Q(q,p) \quad , P = P(q,p)$$

The Hamiltonian function (in phase space) at the transformed variables should equal the Hamiltonian function at the corresponding variables before the transformation:

$$\tilde{H}(Q,P) = \tilde{H}(Q(q,p), P(q,p))$$

$$= H(q,p)$$

Here $\tilde{H}$ is the Hamiltonian that applies to the transformed variables. The question we wish to address is the following: for which kinds of transformations is the form of Hamilton's equations unchanged? That is, when are the equations

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P}, \quad \dot{P} = -\frac{\partial \tilde{H}}{\partial Q}$$

equivalent to

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad ?$$

To answer this question we expand the top two equations using the chain rule to express all derivatives in terms of q and p:

$$\dot{Q} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} = \frac{\partial \tilde{H}}{\partial P}$$

$$= \frac{\partial H}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial P}$$

$$\cancel{\Rightarrow \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} =}$$

$$\frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} = \frac{\partial H}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial P} \quad (A)$$

Similarly, for the $\dot{P}$ equation we get:

$$\frac{\partial P}{\partial q}\dot{q} + \frac{\partial P}{\partial p}\dot{p} = -\frac{\partial H}{\partial q}\frac{\partial q}{\partial Q} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial Q} \quad (B)$$

To eliminate the $\dot{p}$ term we multiply (A) by $\frac{\partial P}{\partial p}$ and (B) by $\frac{\partial Q}{\partial p}$ and subtract:

$$\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\right)\dot{q} =$$

(a)

$$\frac{\partial H}{\partial q}\left(\frac{\partial q}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial q}{\partial Q}\frac{\partial Q}{\partial p}\right)$$

(b)

$$+ \frac{\partial H}{\partial p}\left(\frac{\partial p}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial p}{\partial Q}\frac{\partial Q}{\partial p}\right)$$

(c)

Both (b) and (c) simplify:

$$(b) = \left.\frac{\partial q(Q,P)}{\partial p}\right|_q = 0$$

$$(c) = \left.\frac{\partial p(Q,P)}{\partial p}\right|_q = 1$$

So we recover the Hamilton's equation for the original variables,

$$\dot{q} = \frac{\partial H}{\partial p}$$

provided one condition is satisfied:

$$(a): \quad \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1$$

Going through very similar steps for the other Hamilton's equation we find again that the only condition is equation (a). But equation (a) is simply expressing the fact that the transformation is "locally area-preserving", the Jacobian has determinant equal to unity:

$$\det \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{bmatrix} = 1 .$$

In mechanics we use the term "canonical" for phase-space transformations that locally preserve phase-space area.

Previously we have seen that Hamiltonian flow is also area preserving, so that should give us an example of a canonical transformation:

$$q(0) \longrightarrow q(\Delta t) = q + \Delta t \frac{\partial H}{\partial p} \equiv Q(q,p)$$

$$p(0) \longrightarrow p(\Delta t) = p - \Delta t \frac{\partial H}{\partial q} \equiv P(q,p)$$

Let's verify the transformation is canonical in the limit $\Delta t \to 0$:

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} =$$

$$\left(1 + \Delta t \frac{\partial^2 H}{\partial q \partial p}\right)\left(1 - \Delta t \frac{\partial^2 H}{\partial p \partial q}\right)$$

$$- \left(\Delta t \frac{\partial^2 H}{\partial p^2}\right)\left(- \Delta t \frac{\partial^2 H}{\partial q^2}\right)$$

$$= 1 + \mathcal{O}(\Delta t^2) \checkmark$$

The antisymmetric phase-space derivative product arises in several places, so much so that a new notation was invented.

For arbitrary functions of phase space, $f(p,q)$ and $g(p,q)$, the "Poisson bracket" is defined as

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} .$$

The phase-space transformation functions Q, P are canonical provided

$$\{Q, P\} = 1 .$$

We can use this notation to express some old facts in compact forms. First consider an arbitrary function $A = A(q, p, t)$.

Here is its time derivative:

$$\dot{A} = \frac{\partial A}{\partial q}\dot{q} + \frac{\partial A}{\partial p}\dot{p} + \frac{\partial A}{\partial t}$$

$$= \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial A}{\partial p}\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial A}{\partial t}$$

$$= \{A, H\} + \frac{\partial A}{\partial t}$$

If $A = q$ or $A = p$ we get Hamilton's equations:

$$\dot{q} = \{q, H\} \qquad \dot{p} = \{p, H\}$$

Setting $A = H$ we get energy conservation:

$$\dot{H} = \{H, H\} = 0$$

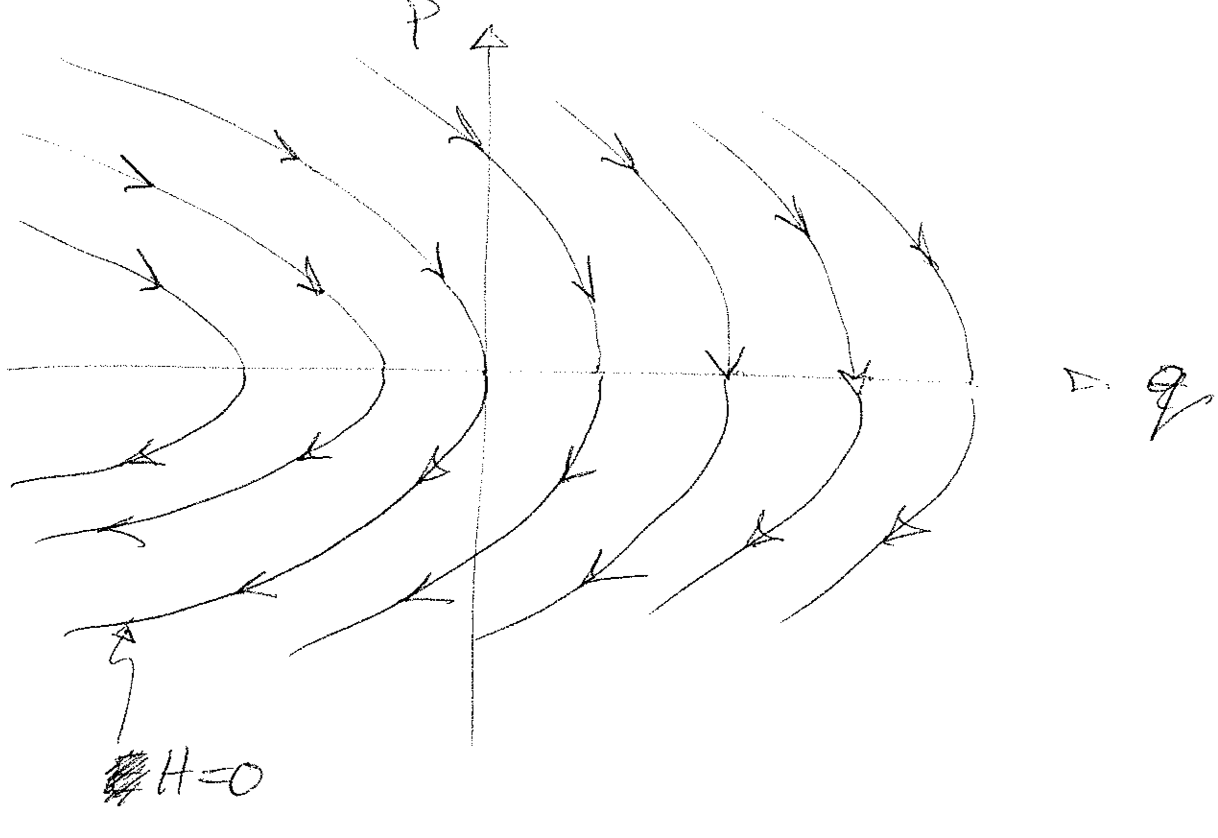
More generally, if a quantity $I(q,p)$ is conserved (and has no explicit time dependence), such as we encounter with Noether's thm, then its Poisson bracket with H vanishes:

$$\dot{I} = 0 \implies \{I, H\} = 0$$

Let's apply a canonical transformation to the problem of a body falling in a uniform gravitational field. We start with the coordinate q and its conjugate momentum p:

$$H = \frac{p^2}{2m} + mgq$$

EOM:
$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m} \\ \dot{p} = -\dfrac{\partial H}{\partial q} = -mg \end{cases}$$



I claim the following is a canonical transformation:

$$Q = q + \frac{p^2}{2m^2 g}$$
$$P = p$$

Let's check, by computing the Poisson bracket:

$$\{Q, P\} = (1)(1) - (0)\left(\frac{p}{m^2 g}\right) = 1 \checkmark$$

The Hamiltonian has a very simple form in terms of the new variables:

$$H(q,p) = \frac{p^2}{2m} + mgq = mgQ$$

$$H'(Q,P) = mgQ$$

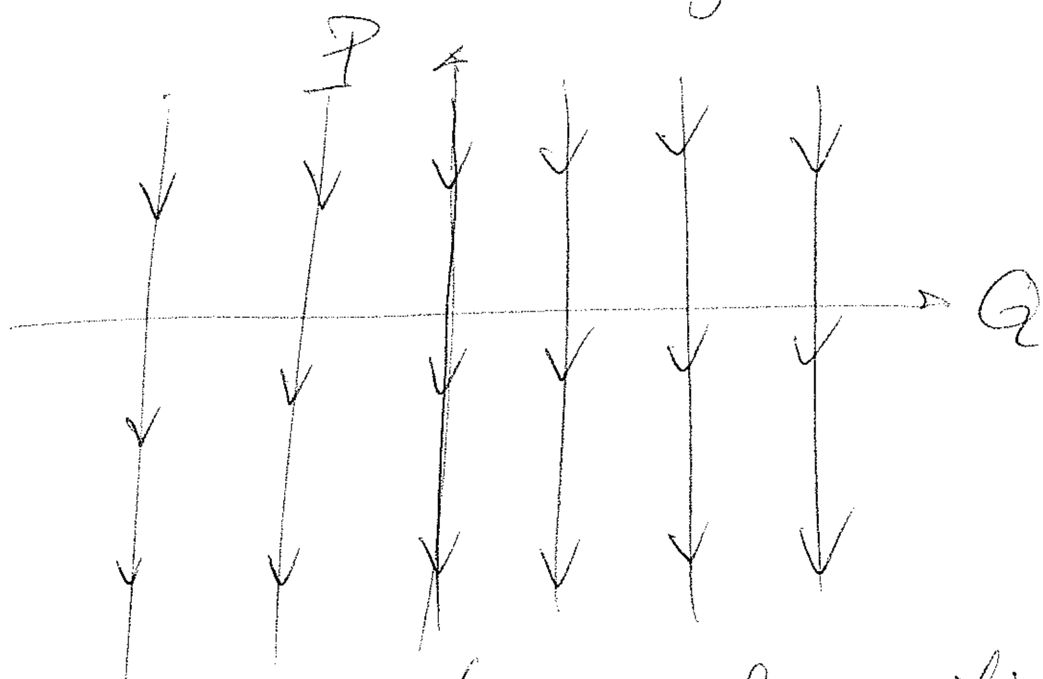
The transformed Hamilton's eq'ns are likewise simple:

$$\dot{Q} = \frac{\partial H'}{\partial P} = 0$$

$$\dot{P} = -\frac{\partial H'}{\partial Q} = -mg$$

solution: $Q = \text{~~const~~}. = Q_0$

$P = -mgt + P_0$



The canonical transformation had, in this case, the effect of straightening the flow

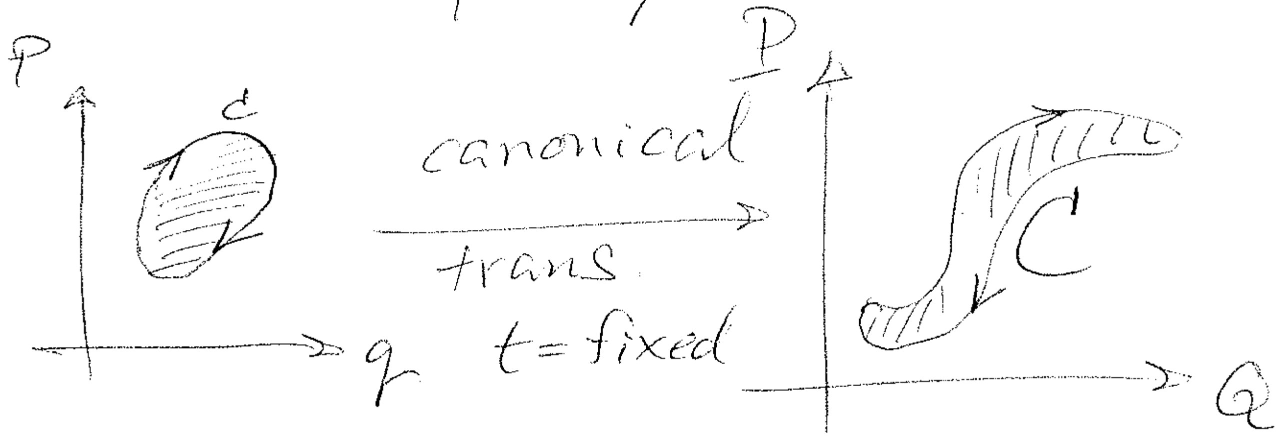
We now extend the definition of canonical transformations to allow for time dependence of the transformation:

$$Q = Q(q, p, t) \quad , \quad P = P(q, p, t)$$

The (infinitesimal) phase-space area preserving property holds for each instant of time:

$$\{Q, P\} = 1 \quad (\text{all } t)$$

However, we will see that with time-dependent canonical transformations it is necessary to also transform the Hamiltonian in order that the form of Hamilton's equations is preserved. To see this, we study the relationship between the areas enclosed by curves in phase space <u>at fixed time</u>, a geometrical property:



Because the transformation is area-preserving, the area enclosed by c equals the area enclosed by the image of c, C':

$$\oint_c p\,dq = \oint_{C'} P\,dQ$$

Re-express this as an integral in just the p-q plane:

$$\oint_c \left(p\,dq - P(q,p,t)\,dQ\right) = 0$$

where $dQ = \frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp$. Since the curve c was arbitrary, this can only be true if we are integrating a perfect differential in the q-p variables:

$$p\,dq - P\,dQ = dF\Big|_t \qquad (A)$$

The notation on the right reminds us that the time is being held fixed. Potentially, F <u>can</u> depend on time, but the differential $dF|_t$ is computed with t held fixed. It will turn out to be ~~useful~~ expedient to define F as a function of three variables like this:

$$F = F(q, Q(q,p,t), t)$$

Then
$$dF|_t = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial Q} dQ|_t \quad (B)$$

where $dQ|_t = \frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp$. Comparing (A) and (B) we get two equations for F:

$$p = \frac{\partial F}{\partial q} \quad , \quad -P = \frac{\partial F}{\partial Q} \quad .$$

The function F is called a "generating function" and the two equations above allows us to relate canonically related phase-space variables, i.e. to "generate" a canonical transformation.

Example : $\quad F = m \dfrac{qQ}{t}$

$$p = \frac{\partial F}{\partial q} = m\frac{Q}{t} \quad , \quad -P = \frac{\partial F}{\partial Q} = m\frac{q}{t}$$

Solve for Q, P in terms of q, p:

$$Q = \frac{p}{m} t \quad , \quad P = -m\frac{q}{t} \quad .$$

Check: $\{Q, P\} = 0 \cdot 0 - \left(-\frac{m}{t}\right)\left(\frac{t}{m}\right)$

$= 1$ ✓