More dynamics in phase space

- four kinds of generating functions

- transformation to "action-angle" variables

Because phase-space area integrals may be expressed in two equivalent ways

$$\oint_C p\,dq = \oint_C (-q\,dp),$$

there are four different ways we could have defined our generating functions:

$$p\,dq - P\,dQ\Big|_t = dF_1\Big|_t$$

$$-q\,dp - P\,dQ_t = dF_2\Big|_t$$

$$p\,dq + Q\,dP\Big|_t = dF_3\Big|_t$$

$$-q\,dp + Q\,dP\Big|_t = dF_4\Big|_t$$

$$F_1 = F_1(q, Q, t) \qquad \frac{\partial F_1}{\partial q} = p \qquad \frac{\partial F_1}{\partial Q} = -P$$

$$F_2 = F_2(p, Q, t) \qquad \frac{\partial F_2}{\partial p} = -q \qquad \frac{\partial F_2}{\partial Q} = -P$$

$$F_3 = F_3(p, P, t) \qquad \frac{\partial F_3}{\partial p} = -q \qquad \frac{\partial F_3}{\partial P} = Q$$

$$F_4 = F_4(q, P, t) \qquad \frac{\partial F_4}{\partial q} = p \qquad \frac{\partial F_4}{\partial P} = Q$$

$$dQ = dQ\big|_t + \frac{\partial Q}{\partial t}\,dt$$

$$dP = dP\big|_t + \frac{\partial P}{\partial t}\,dt$$

$$dF_1 = dF_1\big|_t + \left(\underbrace{\frac{\partial F_1}{\partial Q}}_{-P}\frac{\partial Q}{\partial t} + \frac{\partial F_1}{\partial t}\right)dt$$

$$dF_2 = dF_2\big|_t + \left(-P\,\frac{\partial Q}{\partial t} + \frac{\partial F_2}{\partial t}\right)dt$$

$$dF_3 = dF_3\big|_t + \left(\underbrace{\frac{\partial F_3}{\partial P}}_{+Q}\frac{\partial P}{\partial t} + \frac{\partial F_3}{\partial t}\right)dt$$

$$dF_4 = dF_4\big|_t + \left(Q\,\frac{\partial P}{\partial t} + \frac{\partial F_4}{\partial t}\right)dt$$

$$p\,dq = P\left(dQ - \cancel{\frac{\partial Q}{\partial t}\,dt}\right)$$

$$+\, dF_1 + \cancel{P\,\frac{\partial Q}{\partial t}\,dt} - \frac{\partial F_1}{\partial t}\,dt$$

$$-q\,dp = P\left(dQ - \cancel{\frac{\partial Q}{\partial t}dt}\right) + dF_2$$
$$\qquad + \cancel{P\frac{\partial Q}{\partial t}dt} - \frac{\partial F_2}{\partial t}dt$$

$$p\,dq = -Q\left(dP - \cancel{\frac{\partial P}{\partial t}dt}\right) + dF_3$$
$$\qquad - \cancel{Q\frac{\partial P}{\partial t}dt} - \frac{\partial F_3}{\partial t}dt$$

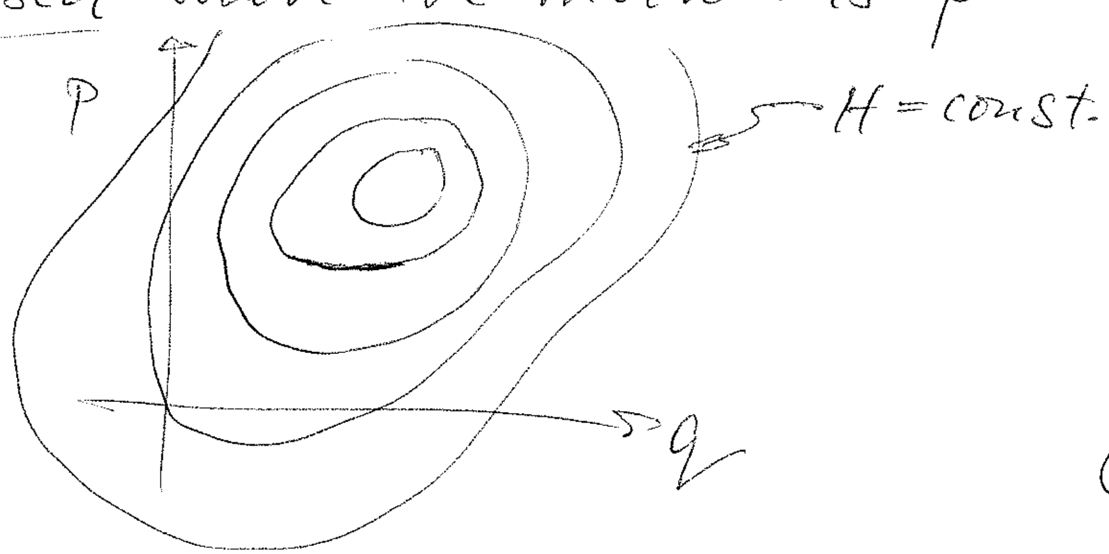
$$-q\,dp = -Q\left(dP - \cancel{\frac{\partial P}{\partial t}dt}\right) + dF_4$$
$$\qquad - \cancel{Q\frac{\partial P}{\partial t}dt} - \frac{\partial F_4}{\partial t}dt$$

We see that in all four cases the form of the action is unchanged provided the Hamiltonian is transformed as

$$H'(Q,P,t) = H(q(Q,P,t),\, p(Q,P,t),\, t) + \frac{\partial F}{\partial t}$$

where F is F_1, F_2, F_3, or F_4.

There is a special choice of phase-space variables that we use for periodic motion. Since the motion follows $H =$ constant contours (for 1 degree of freedom the energy "surface" is just a curve), the contours will be closed when the motion is periodic:



We now define the "action" variable like this:

$$I(q,p) = \left(\begin{array}{l} \text{phase-space area} \\ \text{enclosed by contour} \\ \text{that passess through} \\ q,p \end{array} \right) \Big/ 2\pi$$

I will play the role of the new momentum variable, previously denoted P. By construction, the Hamiltonian is purely a function of I:

$$H = h(I)$$

The new coordinate variable, that I is "conjugate to", will be called θ instead of Q. Hamilton's eq'ns written in terms of the new variables are exceedingly simple:

$$\dot{\theta} = \frac{\partial H}{\partial I} = h'(I)$$

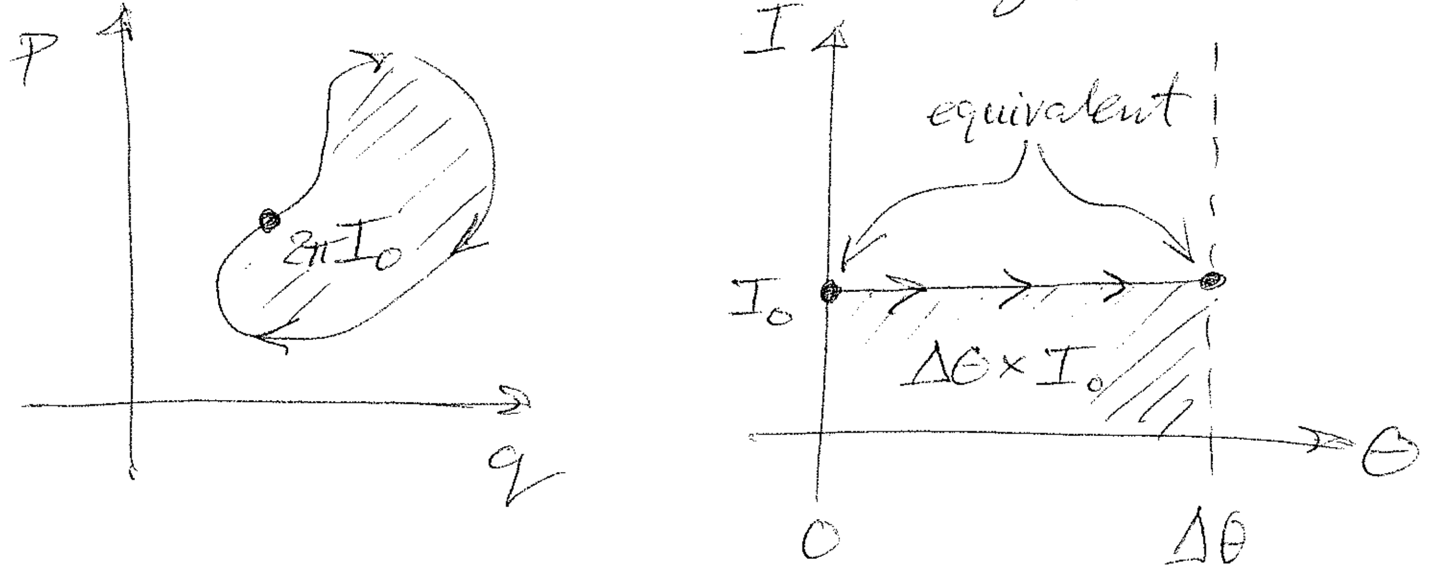
$$\dot{I} = -\frac{\partial H}{\partial \theta} = 0$$

The second equation tells us $I = I_0 =$ constant (something we already knew from $H = h(I) = \text{const.}$).
The first equation is almost as simple:

$$\dot{\theta} = h'(I_0) = \omega_0 = \text{const.}$$

$$\Rightarrow \quad \theta(t) = \omega_0 t + \theta_0$$

The construction of the (θ, I) phase space has periodicity of of the orbits built into its <u>topology</u> by identifying points as equivalent when θ is changed by a fixed amount $\Delta\theta$. The correct value of $\Delta\theta$ follows from our definition of I and the fact that the transformation is canonical (phase-space area preserving):



$$\Rightarrow \quad \Delta\theta = 2\pi$$

Because of the identified points in the θ-I plane, the orbit is actually closed, and the area enclosed is as indicated in the sketch.

Now if T is the period of the orbit for some I_0, then our solution to Hamilton's equations tells us

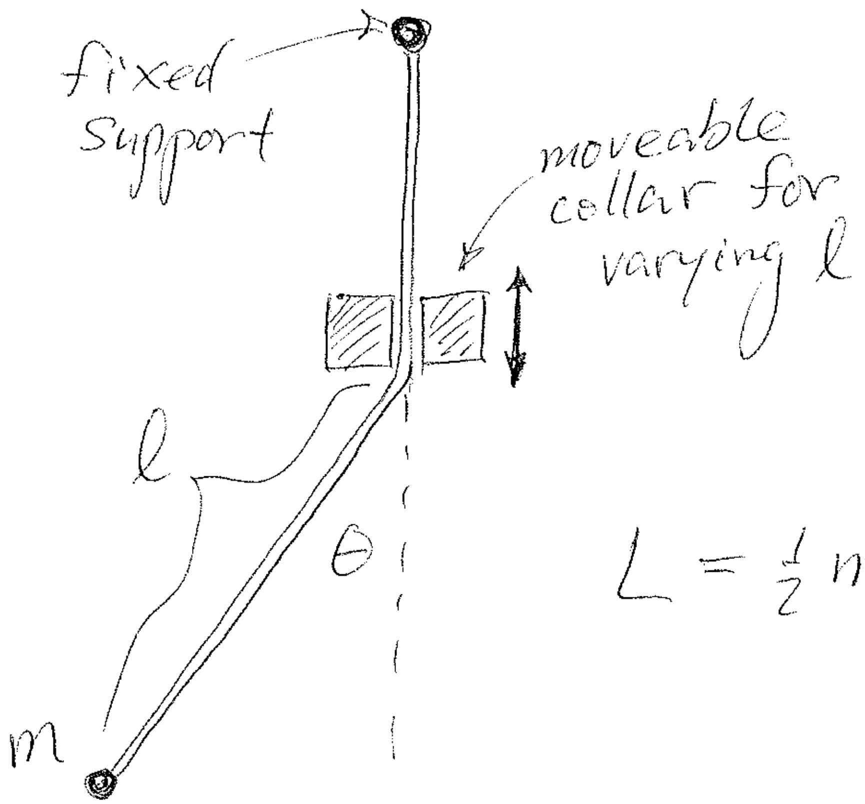
$$\omega_0 T = 2\pi,$$

so

$$h'(I_0) = \frac{2\pi}{T},$$

in words: the derivative of the energy (H) with respect to action ($2\pi I$) is the inverse of the orbit period.

Let's apply the transformation to action-angle variables to the simplest model of periodic motion, the harmonic oscillator. In particular, we will study a pendulum whose string passes through a collar that enables us to vary the length of the string:



$$L = \frac{1}{2} m (l\dot{\theta})^2 - mgl\underbrace{(1-\cos\theta)}_{\simeq \frac{1}{2}\theta^2}$$

$\theta \longrightarrow q$ (so as not to confuse with ~~a~~ action-angle θ)

$$p = \frac{\partial L}{\partial \dot{q}} = ml^2\dot{q}$$

$$H = p\dot{q} - L = \frac{p^2}{2ml^2} + \frac{1}{2} mgl\, q^2$$

define $\omega = \sqrt{g/l}$

$$H = \frac{p^2}{2ml^2} + \frac{1}{2} m\omega^2 l^2 q^2$$

Here is a generating function for the transformation to action-angle variables:

$$F(q, \theta) = \frac{1}{2} m\omega l^2 q^2 \cot\theta$$

(previously, $\theta = Q$). Here are the two equations satisfied by F:

$$p = \frac{\partial F}{\partial q} = m\omega l^2 q \cot\theta$$

$$I = -\frac{\partial F}{\partial \theta} = -\frac{1}{2} m\omega l^2 q^2 \left(\frac{-1}{\sin^2\theta}\right)$$

Solving for q in the second equation:

$$q = \sqrt{\frac{2I}{m\omega l^2}} \sin\theta$$

Substituting this into the first equation:

$$p = \sqrt{2I\, m\omega l^2} \cos\theta$$

Finally:

$$H = \frac{2I\, m\omega l^2}{2ml^2} \cos^2\theta + \frac{1}{2}\frac{m\omega^2 l^2\, 2I}{m\omega l^2} \sin^2\theta$$

$$= \omega I .$$

We see that the harmonic oscillator is especially simple, in the sense that $h(I) = \omega I$, $h'(I) = \omega = \frac{2\pi}{T}$.