25 FEB 2013

Begin the exam when instructed to do so. You have 50 minutes to complete 6 problems of equal weight. All work must be on the exam pages.

Rules: No sources (notes, texts, homework solutions, etc.) or calculators are allowed. A few formulas are given at the end of the exam.

The Cornell Code of Academic Integrity is in effect, as always.

SOLUTION SET

1. How many degrees of freedom do the following two systems have?

ROLLING W/O SLIPPING RELATES 2 PAIRS OF ROT & TRANSL. DOF

   (a) A marble (sphere) rolling without slipping on a surface.

   3 rotational + 2 translation − 2 constraints = $\boxed{3}$ DOF

   (b) A hockey puck (cylinder) sliding (with perfect slipping) on ice. The flat surface of the puck is always parallel to, and in contact with, the ice as it is subject to hockey-stick forces.

   1 rotational + 2 translation = $\boxed{3}$ dof

2. A particle of mass m moves on a plane surface and is subject to a central force derived from the potential $V(r)$. Write down the Lagrangian for the particle using polar coordinates (r, θ) in the plane.

$$\boxed{L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)}$$

3. A system with two degrees of freedom, q_1 and q_2, is described by the Lagrangian

$$L = A\dot{q_1}^2 + Bq_1\dot{q}_2 + C\dot{q_2}^2 + Dq_1^2,$$

where A, B, C and D are constants.

(a) Are any of the conjugate momenta of this system conserved? If so, express them in terms of q's and $\dot{q}$'s.

q_2 does not appear in L

$\Rightarrow P_2 = \frac{\partial L}{\partial \dot{q}}$ is conserved

$$\boxed{P_2 = Bq_1 + 2C\dot{q}_2}$$

(b) Is the Hamiltonian of this system conserved? If so, express it in terms of q's and $\dot{q}$'s.

L is independent of t (explicitly)

$\Rightarrow H$ is conserved.

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$= 2A\dot{q}^2 + Bq_1\dot{q}_2 + 2C\dot{q}_2^2 - L$$

$$= \boxed{A\dot{q}_1^2 + C\dot{q}_2^2 - Dq_1^2}$$

4. For the following items,

A: $\dfrac{\partial L}{\partial q} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) = 0,$ ↙ EULER LAGRANGE COMES FROM $\delta S/\delta q$ VARIATION

B: $\hbar \to 0$ limit of quantum mechanics, → CLASSICAL PATH DOMINATES i.e. extremum of ACTION

C: $\dfrac{\delta S}{\delta q(t)} = 0,$

select the correct chain of logical implication[1]:

- $A \Rightarrow B \Rightarrow C$
- $A \Rightarrow C \Rightarrow B$
- $B \Rightarrow A \Rightarrow C$
- $B \Rightarrow C \Rightarrow A$
- $C \Rightarrow A \Rightarrow B$
- $C \Rightarrow B \Rightarrow A$

5. (a) Calculate the variational derivative $\dfrac{\delta F}{\delta y(x)}$ of the functional

$$F[y(x)] = \int_{x_1}^{x_2} y^2 \, \frac{dy}{dx} \, dx.$$

(b) If your answer to (a) is astonishingly simple (hint!), comment on why that result is not altogether surprising.

(a) $\underbrace{\frac{d}{dx}(y^2)}_{=2yy'} - 2yy' = 0$ IDENTICALLY (not invoking EULER-LAGRANGE!)

(b) $F[y(x)] = \int_{y(x_1)}^{y(x_2)} d\left(\frac{1}{3}y^3\right)$ ← TOTAL DERIVATIVE

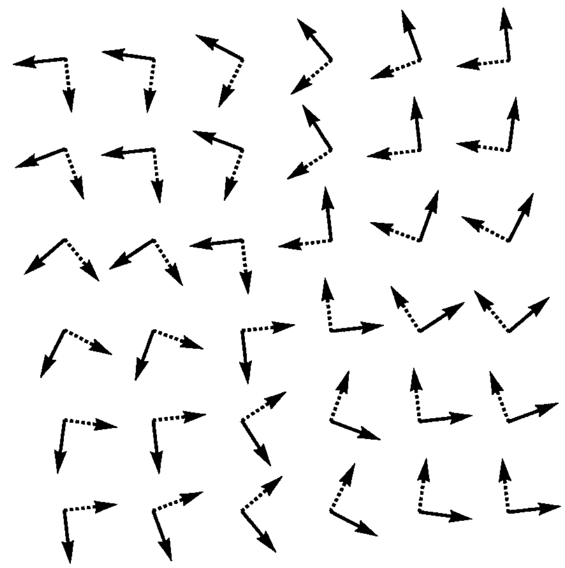
⇒ INTEGRAL IS GIVEN BY $\frac{1}{3}y^3$ EVALUATED @ $y(x_2)$ & $y(x_1)$.

BUT VARIATION @ ENDPOINTS FIXED TO ZERO

⇒ VARIATION OF F VANISHES IDENTICALLY.

[1] $X \Rightarrow Y$ means "X implies Y"

6. A mass m particle moving in the x-y plane is constrained (by a mysterious mechanism) to always **move parallel** to the vector field $\mathbf{a}(x,y)$ shown below (solid arrows):



Also shown, as dashed arrows, is the vector field $\mathbf{b}(x,y) = \hat{\mathbf{z}} \times \mathbf{a}$, which is everywhere perpendicular to $\mathbf{a}(x,y)$.

(a) Which of the following two choices is the correct form of the equations of motion for the position $\mathbf{r}(t)$ of the particle, when the constraint is imposed by Lagrange multipliers $\lambda(t)$?

$$-m\,\ddot{\mathbf{r}} = \lambda\,\mathbf{a} \qquad\qquad -m\,\ddot{\mathbf{r}} = \lambda\,\mathbf{b}$$

constraint force acts to keep variation on allowed direction.

(b) Are the constraints on this particle holonomic or non-holonomic[2]?

FROM PICTURE $\vec{\nabla}\times\vec{b} \neq 0$

$\Rightarrow \vec{b}$ IS NOT CONSERVATIVE.

[2] Fun suggestion for unwinding after the exam: solve for the Lagrange multipliers!

Formulas

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$H = \sum_i p_i \dot{q}_i - L$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

$$S = \int_{t_1}^{t_2} L\,dt$$