P318

Sec 3

8 FEB

- RETURN HW #1 → · PLS $\boxed{\text{BOX}}$ ANSWERS · WRITE NEATLY · STAPLE | we have a new grader!
- "the LAGRANGIAN IS SMARTER THAN US"
- → dof, constraints, (INTEGRABILITY) ← · LAGRANGE MULTIPLIERS
- CALCULUS OF VARIATIONS
- SOME GEOMETRY, "for culture"

"The Lagrangian is smarter than we are"

DON'T GET TOO CAUGHT UP IN COUNTING DEGREES of FREEDOM, the key is:

1. OFTEN YOU NEED FEWER DOF THAN IN THE 116 (Newtonian) APPROACH. <u>IT'S OKAY</u> IF YOU HAVE <u>MORE</u> THAN THE BARE MINIMUM # OF DOF!

2. BUT <u>NEVER</u> EVER USE <u>LESS</u> THAN THE BARE MINIMUM!

↳ there's some "philosophy" in DOF. YOU COULD <u>*POETICALLY*</u> ARGUE THAT THERE'S ONLY ONE DOF SINCE SYSTEM HAS A UNIQUE PATH IN SPACETIME (REALLY: CONFIG SPACE) — @ LEAST CLASSICALLY. (but you can see why STATISTICALLY/QUANTUML'y this isn't nec. true). BUT WE'RE NOT CLEVER ENOUGH TO JUST IDENTIFY THAT ONE DOF TO WRITE $q(t)$ DIRECTLY.
<u>THE POINT</u>: "# DOF" IS ~~~~ NOT A SUPER-RIGID IDEA

$L = T - mgz$

Trivial example : FREE FALLING PARTICLE

~~[illegible]~~ HOW MANY DOF? 3

... BUT REALLY WE KNOW ONLY VERTICAL DIR MATTERS.

WHAT HAPPENED TO OTHER DIRECTIONS?

$\partial L / \partial x = 0 \Rightarrow \partial L / \partial \dot{x}$ is CONSERVED

& $\dot{x}_{init} = 0$, $x_{init} = 0$ } we can forget about it

cf. DEATHNOTE

L didn't care that we were redundant

IN FACT, L was nice enough to tell us that we were redundant.

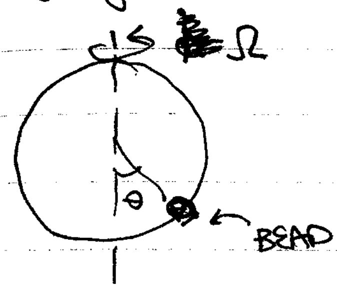
FURTHER: IF OUR REDUNDANCY WAS CONVENIENT (eg NATURAL DESCRIPTION of THE SYSTEM), THEN L is TELLING US ABOUT CONSERVATION LAWS

more on this later in the course

ANOTHER TRIVIAL EX: SYSTEM OF PARTICLES AS A RIGID BODY.

→ ONLY CM (& ROTATIONAL) MOTION MATTERS

Slightly less trivial: SPINNING HOOP PROBLEM ON HW #2.



WHAT IS THIS SYS? I SPIN THE HOOP & LET IT GO? **NO**. THAT IS A DIFFERENT SYSTEM! LET'S DO THAT SYS.

$$L = \tfrac{1}{2} m R^2 \left(\dot\theta^2 + \sin^2\theta \, \dot\varphi^2 \right) - mg(-r\cos\theta) \qquad \dot\varphi_0 = \Omega$$

$$H = \sum_{\dot q_k} \dot q_k \frac{\partial L}{\partial \dot q_k} - L$$

treat as 2 DOF

$$= \underbrace{\tfrac{1}{2} m R^2 \dot\theta^2 + \tfrac{1}{2} m R^2 \underset{\uparrow\ \sin^2\theta}{} \dot\varphi^2}_{T} + \underbrace{mg(-r\cos\theta)}_{V}$$

$V \rightarrow$ INDEP OF φ

$p_\varphi = mR^2 \sin^2\theta \, \dot\varphi$ CONSERVED.

LESSON: IF YOU INTERP. HW PROBLEM AS "SPIN HOOP & LET GO", THEN YOU OVERSIMPLIFIED YOUR CONSTRAINTS

cf <u>HW PROBLEM</u>: ONLY 1 DOF

$$L = \tfrac{1}{2} m R^2 \left(\dot\theta^2 + \sin^2\theta \, \Omega^2 \right) - mg(-r\cos\theta) \qquad (\Omega \text{ const.})$$

$$H = \dot\theta \frac{\partial L}{\partial \dot\theta} - L$$

$$= \tfrac{1}{2} m R^2 \dot\theta^2 - \tfrac{1}{2} m R^2 \sin^2\theta + mg(-r\cos\theta)$$

H is conserved, but <u>not</u> equal to E.

E is <u>NOT</u> CONSERVED $\rightarrow$ ENERGY MUST GO INTO SYST. TO MAINTAIN ANGULAR VELOCITY of THE HOOP!

cf. $\sin^2\theta \, \dot\varphi$ = CONST.

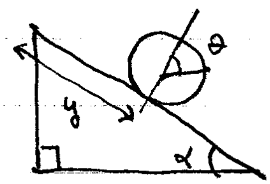
Lagrange Multipliers

how it works
why it works
"magic" w/ HOLONOMIC CONSTR.

HOW IT WORKS (Marion & Thornton eg 7.10)

see LECTURES

Disk rolling down plane (no twisting!)



$T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2$

$U = Mg(l-y)\sin\alpha$ & $\qquad I = \frac{1}{2}MR^2$

constraint: $\boxed{f(y,\theta) = y - R\theta = 0}$

write: $\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda\frac{\partial f}{\partial y}$

MODIFIED EOM:

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \boxed{\lambda\frac{\partial f}{\partial y}} = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \boxed{\lambda\frac{\partial f}{\partial \theta}} = 0$$

& λ UNDETERMINED

→

$M\ddot{y} - Mg\sin\alpha = \lambda$

$I\ddot{\theta} = -\lambda R$

& $y = R\theta \rightarrow \ddot{y} = R\ddot{\theta}$

$I\frac{\ddot{y}}{R} = \lambda R$

$\boxed{-\lambda = \ddot{y}\, I/R^2}$

$$\ddot{y} = \frac{2g\sin\alpha}{3}$$

$$\text{\&}\quad \ddot{\theta} = \frac{2g\sin\alpha}{3r}$$

→ $\boxed{\lambda = \frac{-Mg\sin\alpha}{3}}$

what is this? (DIM. ANALYSIS)
"FRICTION FREE" PRODUCING ROLLING

cf: $\ddot{y} = g\sin\alpha$ (POINT MASS ~~no rolling~~) ↔ SLIDING

Why it works

IN GENERAL
$\sum \lambda_i f_i(\vec{q})$

THIS IS THE SAME AS: $L \mapsto L + \lambda \underbrace{f(q)}_{\text{CONSTRAINT}}$

"auxilliary dof" — UNDETERMINED; CAN THINK OF THIS AS A FAKE DOF
↳ no kinetic term

OBSERVE: ~~THEN~~ EOM w/rt λ JUST GIVES $\boxed{f(q) = 0}$
this is the constraint.

this constraint is communicated to the DOF by modifying their EOM

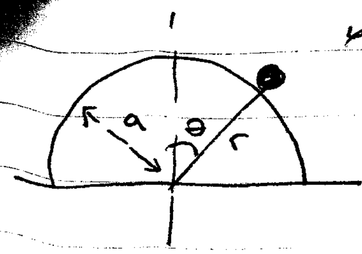
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \lambda \frac{\partial f}{\partial q} = 0$$

ok. I USED DIFF SIGN w/rt PVS EXAMPLE. SIGN ARBITRARY (λ to be det.)

OBSERVE: $\lambda \, \partial f/\partial q$ IS A GENERALIZED force
(q DERIVATIVE TERM)
⇒ PRECISELY THE CONSTRAINT FORCE
w/ MAGNITUDE λ VARYING AS NECESSARY
FOR THE PROBLEM.

↳ look @ this & meditate & you'll see that
$\lambda \, \partial f/\partial q$ PRECISELY CANCELS THE MOTION
WE DON'T ALLOW.
try trivial ex: particle on a table

CLASSIC EXAMPLE



← when does it fall off? (LOSE CONTACT)

when constraint force $= 0$

(ξ afterward our $L + \lambda f$ LAGRANGIAN IS INVALID $\to \lambda = 0$ afterward)

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mg(-r\cos\theta) \pm \underbrace{\lambda(r-a)}_{f}$$

↑ choose +

EoM: r: $m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta + \lambda = 0$

θ: $mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0$

λ: $r - a = 0$

observe: by construction EoM for λ is algebraic

ξ it also implies (IN THIS CASE) $\dot{r} = \ddot{r} = 0$.

kills radial dir. forces

"conservation" or r

$\Rightarrow$ ~~$ma\dot{\theta}^2 = mg\cos\theta$~~

$-am\dot{\theta}^2 + mg\cos\theta + \lambda = 0$

$ma^2\ddot{\theta} - mga\sin\theta = 0 \to \ddot{\theta} = \frac{g}{a}\sin\theta$

$$\frac{d}{dt}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

$\int_0^{\dot{\theta}(\theta)}$ ξ $\int_0^{\theta}$

$$\Rightarrow \int \ddot{\theta}\, d\theta = \int \dot{\theta}\, d\dot{\theta} = \int \frac{g}{a}\sin\theta\, d\theta$$

$$\Rightarrow \frac{1}{2}\dot{\theta}^2 = -\frac{g}{a}\cos\theta + g/a$$

IN:

$$am\left(-\frac{2g}{a}\cos\theta + \frac{2g}{a}\right) + mg\cos\theta + \lambda = 0$$

$$\Rightarrow (3mg\cos\theta - 2mg) = -\lambda$$

$\lambda = 0$ WHEN $\boxed{\cos\theta = \frac{2}{3}}$

Remarks

- SEE BOOK FOR "PENNY ROLLING DOWN A WEDGE" DONE USING LAGRANGE MULTIPLIERS

$\downarrow$

Magic! I thought this wasn't holonomic!!

MT 253

CONSTRAINT TAKES FORM $\boxed{A\dot{x} + B = 0}$

not holonomic

BUT: if $\begin{cases} A = \partial f/\partial x \\ B = \partial f/\partial t \end{cases}$

CONSTRAINT IS: $\underbrace{(\partial f/\partial x)\frac{dx}{dt} + \partial f/\partial t}_{} = 0$

$$\dot{f}(x,t) = 0$$

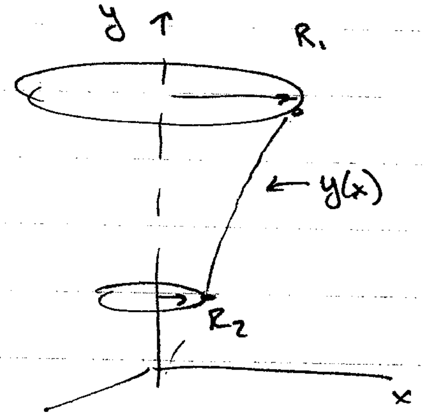
$\Rightarrow \boxed{f(x,t) - \text{CONST} = 0}$

this IS HOLONOMIC

THAT is HOW H&F SOLVE PENNY PROBLEM W/ LAGRANGIAN METHOD.

REMARKS ON VARIATIONAL CALCULUS

eg. (M&T ex 6.3)



What is the shape of a soap bubble between these two rings?

ROT SYM: this is a surface of revolution w/rt $y(x)$

BUBBLE: MINIMIZE SURFACE AREA

$$dA = \underset{\uparrow \text{RADIAL DIST.}}{2\pi x} \underset{\uparrow\, ds,\ \text{ARCLEN.}}{\sqrt{dx^2 + dy^2}}$$

$$A = 2\pi \int_{x_1}^{x_2} \underbrace{x\,(1+(y'(x))^2)^{1/2}}_{F(y,x)}\, dx$$

$F(y, x) \leftarrow$ where $y = y(x)$

cf $q = q(t)$

PLUG INTO EULER-LAGRANGE:

$$\frac{d}{dx}\left[\frac{xy'}{\sqrt{1+y'^2}}\right] = 0 \Rightarrow \frac{xy'}{\sqrt{1+y'^2}} = \text{CONST.} = a$$

$\uparrow\ \partial F/\partial y'$

$$y' = \frac{a}{\sqrt{x^2 - a^2}}$$

$$\Rightarrow \boxed{y = a\cosh^{-1}\left(\frac{x}{a}\right) + b}$$ w/ a & b to fit BNDY.

Thoughts on Variational calculus

WEEK 1-2

EULER-LAGRANGE = NEWTON EQ. ← "3 LAWS"

WEEK 3

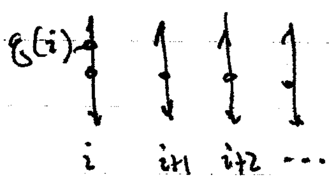
EULER LAGRANGE ⊃ PRINCIPLE of LEAST ACTION

minimize a scalar quantity

WE NOW HAVE AN OBJECT, THE ACTION, WHICH IS A FUNCTIONAL. WE'RE BASICALLY DOING CALCULUS FOR AN INFINITE NUMBER of COUPLED VARIABLES.

↳ think of $\underset{\uparrow_{\text{INDEX}}}{q(t)} \sim q_t$

eg: system of springs, coupled HO, mattress, ...



w/ COUPLING $\sim (q(i) - q(i+1))$

$\sim q'$ → KINETIC TERM

GIVES AN IDEA of HOW TO MAKE THIS RELATIVISTIC

$$S = \int dt\, L \longrightarrow \int d^4x\, \mathcal{L}$$

DETAILS: eg Arnold, Jose & Saletan, etc.

~~of the DAY~~

IS A FUNCTION ON TM ($L: TM \to \mathbb{R}$)
TM: n-DIM

EULER-LAGRANGE EQNS ARE A SET OF $\boxed{2n}$ 1ST O DIFF EQ ON $\boxed{2n}$ DIM PHASE SPACE
PHASE SPACE: TM

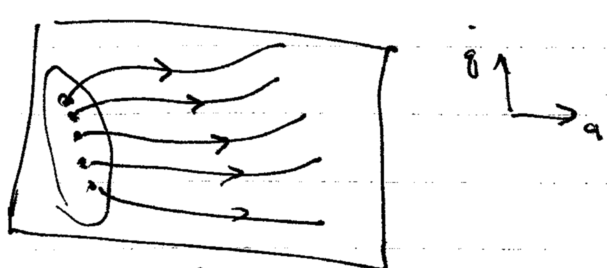
$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} &= 0 \\ \dot q &= \frac{d}{dt} q \end{aligned}\right\} ?$$

holonomic ⟺ integrability

WE WILL MAKE THIS MANIFEST W/ HAMILTONIAN FORMALISM (but can see hints using E, P, ...)

1ST O ODE: ∃ SOLUTION & IT IS UNIQUE (PICCARD THM OR SOMETHING)
GIVEN A SET OF <u>INIT CONDITIONS</u>, YOU CAN GENERATE A FLOW:

eg $q(0)$, $\dot q(0)$ OR EVEN THINGS LIKE E



time evolution of system

IN OTHER WORDS, we <u>integrate</u> to get a flow
FOR SUFFICIENTLY NICE INIT CONDITIONS, YOU CAN <u>FIBRATE</u> TM W/ THESE FLOWS.

"HOLONOMIC" CONSTRAINTS ⟷ notion of this integrability
the flow lines (vector field) <u>can</u> BE INTEGRATED ("draw lines through")