P3318 SECTION 9 5 April

Prelim 2 review

~~[illegible]~~ How to prep

- review your problem sets!
- do similar practice problems

↓

Don't worry about technical details (EXAM WONT TEST ABILITY TO DO CALCULUS)

YOU WILL HAVE A LIST OF EQNS; won't have to memorize. SEE LECT SEC NOTES

What have we done since Prelim 1?

[GRAVITATIONAL] 2 Body Problem

$$L = \frac{1}{2}m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2 - V(|\vec{r}_1 - \vec{r}_2|)$$

6 DOF? gtfo.

CAN REDUCE THIS TO ONE DOF

↳ SEPARATE cm MOTION FROM RELATIVE MOTION

$$L = \underbrace{\frac{1}{2}(m_1+m_2)|\dot{\vec{R}}|^2}_{\text{free particle}} + \frac{1}{2}\mu|\dot{\vec{r}}|^2 - V(|\vec{r}|)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

- MOTION CONSTRAINED TO A PLANE
- ANGULAR MOMENTUM CONSERVED, l

left w/ 1 dof: r

Remark: $\ell = \text{const}$ from ISOTROPY of L via Noether.

Where did "motion on a plane" come from?

$$\frac{d}{dt}(\dot{\vec{r}}\times\vec{r}) = \ddot{\vec{r}}\times\vec{r} = 0$$

EOM: $\ddot{\vec{r}} \propto \vec{r}$

CULTURE: YOU CAN ALSO DERIVE THIS FROM NOETHER!
Runge-Lenz vector is constant (for $1/r$ POT.)
"HIDDEN SYMMETRY" - $SO(4)$ ~ 4D ROTATIONS!

OBTAIN $L_{\text{rel}} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

$$\ell = \frac{\partial L}{\partial\dot{\theta}} = \mu r^2\dot{\theta} = \text{const.}$$

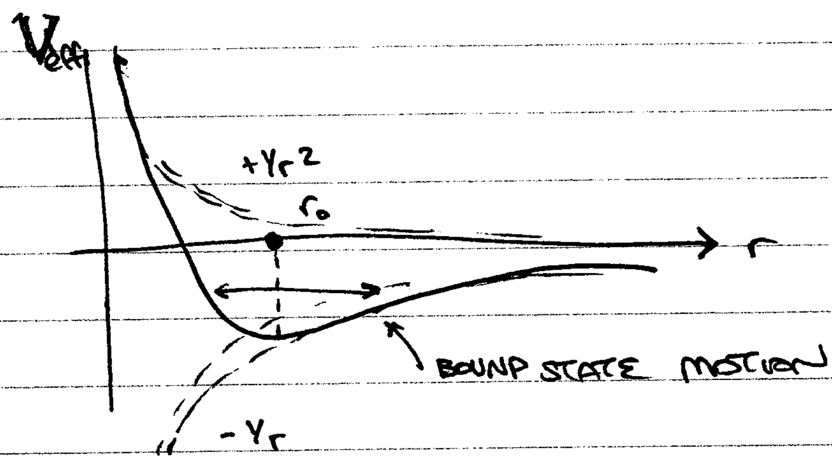
$$\text{L.EOM:}\quad \underbrace{\mu r\dot{\theta}^2}_{\frac{\ell^2}{\mu r^3}} - V'(r) = \mu\ddot{r}$$

$$\Rightarrow \boxed{\mu\ddot{r} = \underbrace{\frac{\ell^2}{\mu r^3}}_{\text{centrifugal}} - V'(r)}$$

$$-\frac{\partial V_{\text{eff}}}{\partial r}$$

$\ell^2/2\mu$ $\quad$ eg $GM_1 m_2$

$$V_{eff}(r) = \frac{B}{r^2} - \frac{A}{r}$$



SOLVING : L formalism → 2nd O ODE ... HARD

find & use constants of motion!

ℓ, defined above

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} - \frac{A}{r}}_{V_{eff}(r)}$$

CHANGE INDEP VAR FROM $t \to r$

$$\dot{r}^2 \to \left(\frac{dr}{d\theta}\underbrace{\frac{d\theta}{dt}}_{\ell/\mu r^2}\right)^2$$

$$E = \frac{1}{2}\frac{\ell^2}{\mu r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{\ell^2}{2\mu r^2} - \frac{A}{r}$$

CLEVER TRICK: $r(\theta) = 1/u(\theta)$

$$\frac{dr}{d\theta} = \frac{-1}{u^2}\frac{du}{d\theta}$$

$$E = \frac{l_z^2}{2\mu}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - Au$$

USE $\dot{E} = \partial E/\partial\theta \, \dot{\theta} = 0$

$$\boxed{\frac{d^2u}{d\theta^2} + u = A\frac{\mu}{l^2}}$$

linear θ: $\left(\frac{d^2}{d\theta^2} + 1\right)u = \text{const.}$

$$u(\theta) = \frac{A\mu}{l^2} + u_0\cos(\theta - \theta_0)$$

SPECIFIC SOL   SOL. of HOMOG. EQ.

periodic in θ
(manifest in u variables)
↳ this is why the PURE $1/r$ ATTRACTIVE POTENTIAL IS ALSO PERIODIC & STABLE AGAINST PERTURBATIONS. OF GRAV QUADRUPOLE

nicer form

$$u(\theta) = r(\theta)^{-1} = \frac{1}{r_0}\left(1 + \varepsilon\cos(\theta - \theta_0)\right)$$

$r_0 = l^2/A\mu$ ECCENTRICITY $= \dfrac{u_0}{A\mu/l^2}$

BOUND ORBIT: $0 \leq \varepsilon \leq 1$

$$r_{max} = \frac{r_0}{1-\varepsilon} \qquad r_{min} = \frac{r_0}{1+\varepsilon}$$

can show that you really get an ellipse.

↳ Use: $r_0 = r(1+\varepsilon\cos\theta)$

$r\cos\theta = x$

reorrange & write in form $x^2+y^2 \equiv r^2 = (r_0-\varepsilon x)^2$

FINALLY: $(1-\varepsilon^2)(x-x_0)^2 + y^2 = r_0^2 + (1-\varepsilon^2)x_0^2$

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \begin{cases} a = r_0/1-\varepsilon^2 \\ b = r_0/\sqrt{1-\varepsilon^2} \end{cases}$$

Kepler's law ①: sweep out equal area in equal time

$$\Delta A = \left(\frac{\Delta\theta}{2\pi}\right)\pi r^2 = \underbrace{\frac{1}{2} r^2 \dot{\theta}}_{l/2\mu \,=\, \text{CONST}} \Delta t$$

Kepler's law ②: $(\text{PERIOD})^2 \sim (\text{SEMI MAJOR AXIS})^3$

1st law → $A = \frac{l}{2\mu}\tau$ $\qquad b^2 = a^2(1-\varepsilon^2) = a r_0$ (def a)

ELLIPSE → $A = \pi ab$

$$\Rightarrow A^2 = \left(\frac{l}{2\mu}\right)^2 \tau^2 = \pi^2 a^2 \cdot a r_0$$

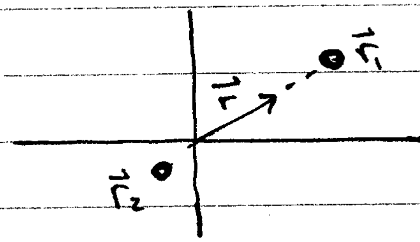
$$= \pi^2 r_0 a^3$$

Actual ORBITS

$\vec{r}$ is relative separation

EVENTUALLY WE WANT ACTUAL ORBITS $\vec{r}_1$ & $\vec{r}_2$

$$\vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r} \qquad \vec{r}_2 = \frac{-m_1}{m_1+m_2}\vec{r}$$



in this pic, which one is massive?

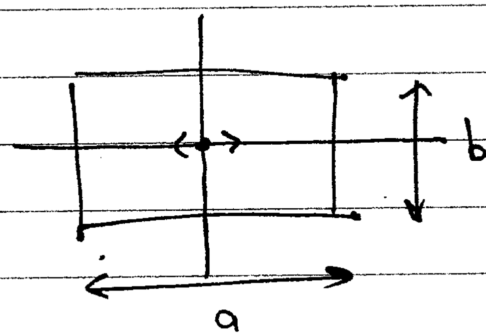
ASSUMING $\vec{R}_{cm} = 0$

GIVEN A TRAJECTORY (ORBIT) $\vec{r}(\theta)$, CAN WE UNDERSTAND $\vec{r}_{1,2}(\theta)$?

1. draw $\vec{r}(\theta)$: $\varepsilon \rightarrow$ Major & minor axes a, b

$a = r_0/1-\varepsilon^2$, $b = r_0/\sqrt{1-\varepsilon^2}$

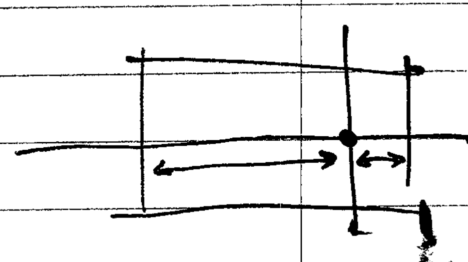
GIVES 'BOUNDING BOX'



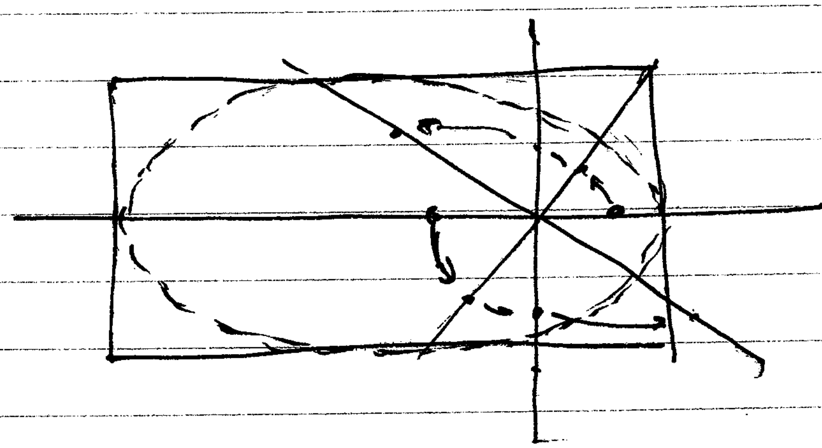
but where to put origin?

2. NEED ~~r_{max} & r_{min}~~ $r_{max,min} = \dfrac{\ell^2/\mu A}{1 \pm \varepsilon}$

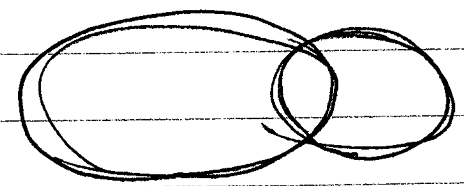
So: $\dfrac{r_{max}}{r_{min}} = \dfrac{1+\varepsilon}{1-\varepsilon}$



3. $\vec{r}_1$ & $\vec{r}_2$ as a function of $\vec{r}$
(mass ratios)



END UP w/

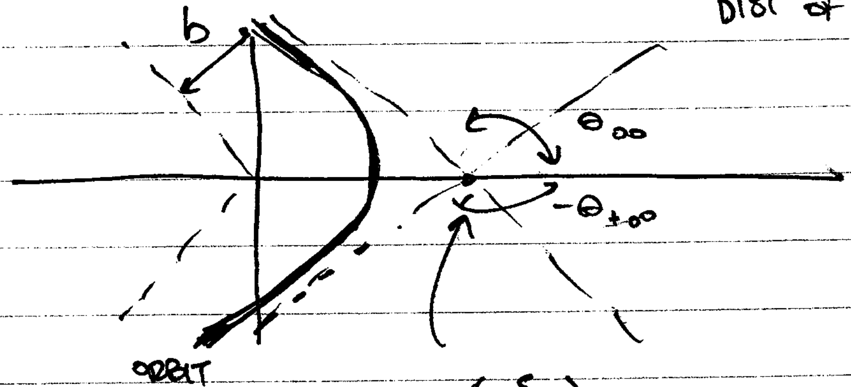


"BREAK UP" SOLUTION: HYPERBOLAE

↳ some manipulations, $\varepsilon > 1$
END UP w/ $\quad 1 = \dfrac{(x-x_0)^2}{a^2} - \dfrac{y^2}{b^2}$

2 DISCONNECTED BRANCHES: ONLY ONE PHYSICAL
(other one came from SQUARING ORBIT EQ)

impact param. SHIFT OF ASYMPTOTES
DIST OF CLOSEST APPROACH



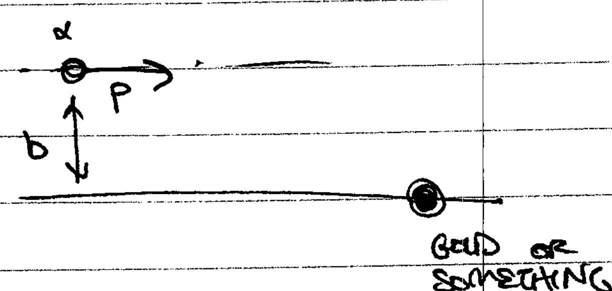
$$x_0 = \left(\frac{\varepsilon}{\varepsilon^2-1}\right) r_0$$

$$r(\theta) = \frac{r_0}{1+\varepsilon\cos\theta} \Rightarrow \cos\theta > -1/\varepsilon$$

$r > 0$, $r_0 > 0$

same eq as ellipse
diff sign for ε!

eg rutherford:



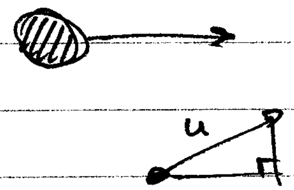
$$\ell = bp = b\mu v_\infty$$

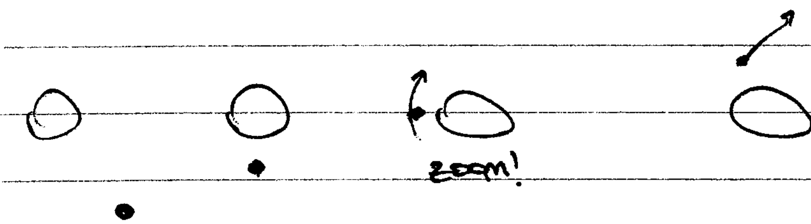
also: $E = \frac{1}{2}\mu v_\infty^2$

$$\left.\right\} \frac{\ell^2}{E} = 2\mu b^2$$

can write in terms of ε
FIND: b really is "b" of hyperbola eq.

Grav. Assist: sattelite scattering





<u>Noether</u>: $L(q, \dot{q}, t) = L(\dot{q}, t)$

$\rightarrow p_q = \partial L / \partial \dot{q}$ conserved.

FORMALIZE: $q \rightarrow Q(s)$

symmetry: $\frac{d}{ds} L(Q(s), \dots)\Big|_{s=0} = 0$

$\hookrightarrow I = p \frac{\partial Q}{\partial s}\Big|_{s=0}$ conserved

$$\frac{d}{ds} L = \underbrace{\frac{\partial L}{\partial Q}}_{\text{EOM}} Q' + \frac{\partial L}{\partial \dot{Q}} \dot{Q}'$$

→ WRITE AS $\frac{d}{dt}(--)$

YOUR HW: SAW MORE GENERALLY THAT YOU CAN HAVE $L \rightarrow L + \underbrace{dF/dt}_{\text{canonical trans.}}$!

<u>MUXIN'S QUESTION</u>: what about SCALE TRANSFORMATIONS?

$$q \mapsto e^{\alpha} q = \mu q \equiv Q$$
$$p \mapsto e^{\beta} p = \nu p \equiv P$$

IS it canonical? $\{Q, P\} = \mu\nu \neq 1$

so what?

IS THERE A TRANSF of H THAT LEAVES THE PHYSICS INVARIANT?

yes: from structure of H. EOM.

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = \frac{-\partial H}{\partial q}$$

$$\to \mu q \qquad \to \frac{1}{\nu}\frac{\partial H}{\partial p} \qquad \to \nu p \qquad \to \frac{1}{\mu}\left(\frac{-\partial H}{\partial q}\right)$$

$\Rightarrow$ if $H \to \mu\nu H$, then this is a SYMMETRY of THE <u>EQUATIONS of MOTION</u>.

further: $H = p\dot{q} - L$

$\to \mu\nu H \qquad \to \mu\nu\, p\dot{q} \qquad \Rightarrow \boxed{L \to \mu\nu L}$

not a sym of L (in gen.)

compare to our discussion of Generating functions: PHYS. UNCHANGED IF

$$L \mapsto \lambda L + \frac{dF}{dt}$$

RESCALING p & q (CHOICE OF <u>UNITS</u>)

CANONICAL TRANS.

example of "DYNAMICAL SYM" (sym of EOM, not L)

also Galilean sym: $\dot{q} \to \dot{q} + v_0$

<u>culture</u>: types of symmetries / cons. quantities

1. sym. of L $\longrightarrow$ discrete (eg PARITY)
   $\searrow$ continuous : GIVES NOETHER CONS. LAW

2. sym of EOM, not L $\longrightarrow$ "dynamical"
   eg $q \to \lambda q$, $\dot{q} \to \dot{q} + v_0$

3. REDUNDANCY of L $\longrightarrow$ GAUGE sym. (eg E&M)
   ($\sim$ phase in QM) OF A STATE
   $\hookrightarrow$ gives <u>FORCES</u>

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4. topologically conserved quantity $\longleftarrow$ knots, mobius strip

Hamiltonians

Why? $\longrightarrow$ 1st ∂ EOM
PHASE SPACE

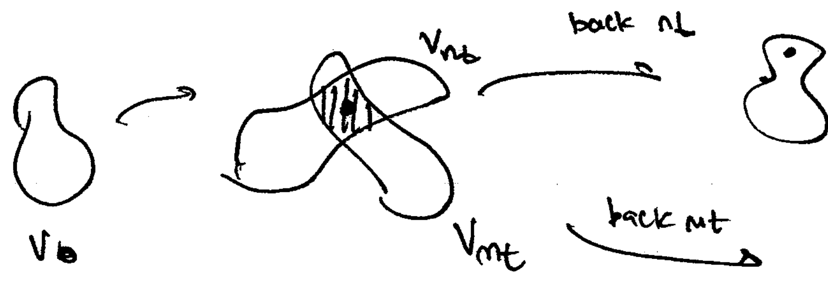
$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\partial H/\partial q$$
"symplectic"

you've done many problems on this recently, so we won't harp on the basics:

INTERESTING RESULTS

- Liouville's thm : BLOB of initial ~~condition~~ conditions may change shape in phase space, but won't change density.
  ↳ CONSISTENT w/ QM: $\Delta p \Delta x \gtrsim \hbar$

- Poincaré recurrence : GIVEN FINITE ACCESSIBLE PS
  very constraining



resolution w/ entropy?

Canonical transformations

Why: transform coordinates (q, p, H) in a way which can make H simpler — eg by making vars. cyclic.

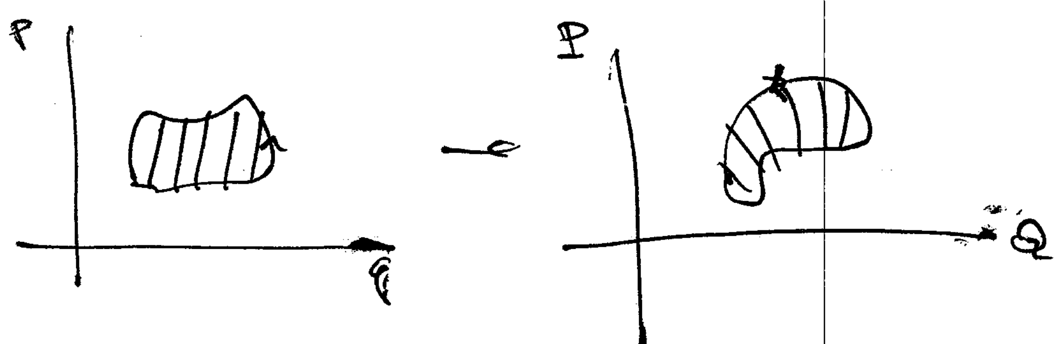
↳ we saw the end result of this today w/ Action-Angle vars.

CANONICAL TRANSF. PROPERTIES

recall: $\dot{f} = \frac{\partial f}{\partial t} + \{F, H\}$

- PS Area preserving

↳ $\underbrace{\det\left|\frac{\partial(P,Q)}{\partial(p,q)}\right|}_{\text{jacobian}} = 1 = \{P, Q\}_{p,q}$



$\oint p\,dq$ OR $-q\,dp$

ORIENTATION

$\oint P\,dQ$ OR $-Q\,dP$

How to construct canonical transf

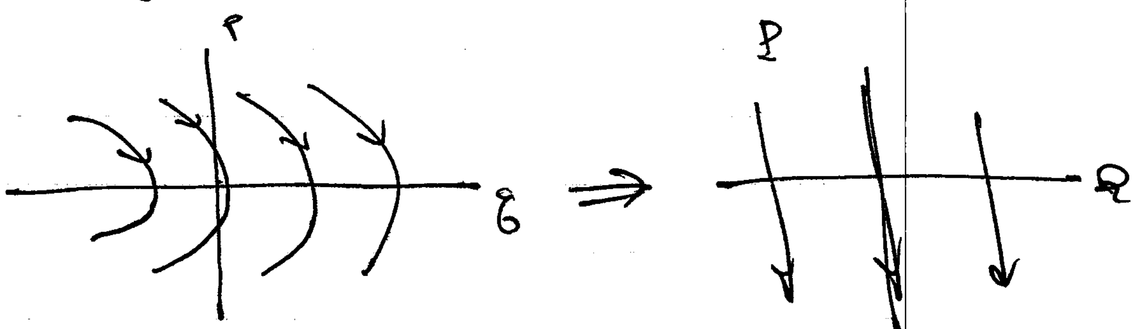
Generating functions

$F \rightsquigarrow$ eg $F_1(q, Q)$ ↖ if t dep, then $H \to H'$

(& 3 others)

examples

falling particle:



see LEC NOTES

harmonic osc. (see notes)

$H = p^2 + q^2 \longleftrightarrow H = P$ ↙ of ACTION ANGLE!

UNDERSTAND: why ~~&~~ FIXING TIME WAS IMPORTANT

$$\oint p\,dq - P\,dQ\big|_t = dF\big|_t$$

encodes time dep of transformation
(ADDITIONAL TIME DEP. BEYOND THAT
of the SENSIBLE COORD $q(t)$)

~~[illegible]~~ SUGGESTIONS

- If you're not sure, try to show some understanding! (COARSE GRAINING of POINTS)
- GO THROUGH HW - do them up to the "COMPUTATIONALLY ANNOYING" step