P881B Section 11 19 Apr.

- HW HINTS
- MOVING FRAMES
- APPLICATION: the falling cat
- SPS TALK

Integrals

- Assuming[{A ∈ Reals, A > 0},
  Integrate[$\frac{\sqrt{A - x^2}}{1+x^2}$, {x, $-\sqrt{A}$, $\sqrt{A}$}]]

- ArcSin[$\sqrt{\frac{A+1}{A}}$ Sin k]

  I couldn't figure it out

  ONE WAY TO CHECK:

  $I = \int(\cdots)$

  CHECK THAT $\frac{d}{dq}$ ArcSin$(\sim) = (\cdots)$.

"Dreibeins" $\rightarrow$ the moving frame

Notation: SPACE FRAME K' (or $\tilde{K}$, etc)
BODY FRAME K

BASIS VECTORS: $\underline{\tilde{e}}_i \cdot \underline{\tilde{e}}_j = \delta_{ij}$
↑
Metric

$$\underline{e_i(t)} \cdot \underline{e_j(t)} = \delta_{ij}$$

$$\underset{\substack{\uparrow \\ \text{coord free} \\ \text{object}}}{\underline{v}} = \underset{\substack{\uparrow \\ \text{t-dep}}}{\tilde{v}_i(t)}\,\underline{\tilde{e}}_i = v_i\,\underline{e}_i(t)$$

IN FACT, TO GO FROM $\underline{\tilde{e}}_i \rightarrow \underline{e}_j(t)$, JUST DO A ROTATION

$$\underline{e_i(t)} = R_{ij}(t)\,\underline{\tilde{e}}_j$$

$$R_{ij}(t) = \underline{e}_i \cdot \underline{\tilde{e}_j(t)} \leftarrow |i\rangle\langle\tilde{j}(t)|$$

So:

$$v_i\,\underline{e}_i(t) = \underbrace{v_i R_{ij}(t)}\,\underline{\tilde{e}}_j$$

$$= \tilde{v}_j(t)$$ ← note: active vs passive transformation

TODAY IN CLASS: ANGULAR VELOCITY

$$\underline{r}(t) = \tilde{r}_i(t)\,\underline{\tilde{e}}_i = r_i\,\underline{e_i(t)}$$

$$\dot{\underline{r}}(t) = \dot{r}_i(t)\,\underline{\tilde{e}}_i = r_i\,\underline{\dot{e}_i(t)}$$

$$= r_i\,\dot{R}_{ij}(t)\,\underline{\tilde{e}}_j$$

$$= r_i \boxed{\underbrace{\dot{R}_{ij}(t)\left[R^{-1}(t)\right]_{jk}}_{\omega_{ik}}}\underbrace{\left(R_{k\ell}\,\underline{\tilde{e}}_\ell\right)}_{\underline{e}_k}$$

(USED: $1 = R^T R$)

OBSERVE: ω_{ik} is antisymmetric

$$\hookrightarrow (R^T)_{ij} R_{jk} = \delta_{ik}$$
$$R_{ji} R_{jk} = \delta_{ik} \quad \text{\& take time deriv.}$$

then write angular velocity PSEUDO-VECTOR

$$\omega_i = \tfrac{1}{2}\varepsilon_{ijk}\,\omega_{jk}\,, \quad \text{BODY FRAME}: \ \omega = \omega_a\,\underline{e_a}(t)$$

$$\rightarrow \frac{d\underline{e}_i}{dt} = \omega_{ij}\,\underline{e}_j = \underline{\omega} \times \underline{e}_i$$

FOR CULTURE — BUT TO CLARIFY CONCEPTS.

We defined angular velocity w/rt how body frame basis vectors change.

ALTERNATELY: WHAT ABOUT SPACE FRAME COORDINATES, $\tilde{r}_i(t)$?

$$\tilde{r}_i(t) = r_j R_{ji}(t)$$

$$\begin{aligned}
\dot{\tilde{r}}_i(t) &= r_j \dot{R}_{ji}(t) \\
&= \tilde{r}_j (R R^{-1})_{jk} \dot{R}_{ki}(t) \\
&= \tilde{r}_j \underbrace{(R^{-1} \dot{R})_{ji}}_{\Omega_{ji}}(t)
\end{aligned}$$

SO: INSTANTANEOUS ANG. VEL: $\omega_{ij} = (\dot{R}(t) R^{-1}(t))_{ij}$

"CONVECTIVE" ANG. VEL: $\Omega_{ij} = (R^{-1}(t) \dot{R})_{ij}$

Path ordering - interesting thing about matrices
↳ really important in QM

$$\omega = \frac{dR}{dt} R^{-1}$$

↑
ah! I can solve this!
think: $\omega\, dt = \frac{dR}{R}$

$$\Rightarrow \quad R(t) = \exp\left(\int_0^t \omega(t')\, dt'\right) \qquad (?)$$

but: R & ω are matrices. much more subtle!
what does exp(M) mean?

$$e^M = 1 + M + \tfrac{1}{2}M^2 + \cdots$$

LET'S CHECK TIME DERIVATIVES of (?)
FIRST TWO TERMS ARE FINE, satisfy $\frac{dR}{dt} = R\omega$
BUT 2ND O:

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^t \omega(t')\,dt'\right)^2 \stackrel{?}{=} \frac{1}{2}\omega(t)\left(\int_0^t \omega(t')\,dt'\right) + \frac{1}{2}\left(\int_0^t \omega(t')\,dt'\right)\omega(t)$$

WTF?.

SECOND TERM is FRUSTRATING!

ORIGIN: CHAIN RULE + $[\omega(t), \omega(t')] \neq 0$ for $t \neq t'$

CANNOT COMMUTE THESE MATRICES.

SOLUTION: DEFINE PATH ORDERING

$$R(t) = \mathcal{P}\exp\left(\int_0^t \omega(t')\, dt'\right)$$

$0 < t' < t'' < t$

$$= 1 + \int_0^t \omega(t')\, dt' + \int_0^{t''}\int_{t'}^{t} \omega(t'')\,\omega(t')\, dt'\, dt'' + \dots$$

<u>result</u>: matrices that appear @ later times appear to the <u>left</u>.

$$2^{ND}\ \mathcal{O} \sim \omega(t)\int_0^t \omega(t')\, dt'$$

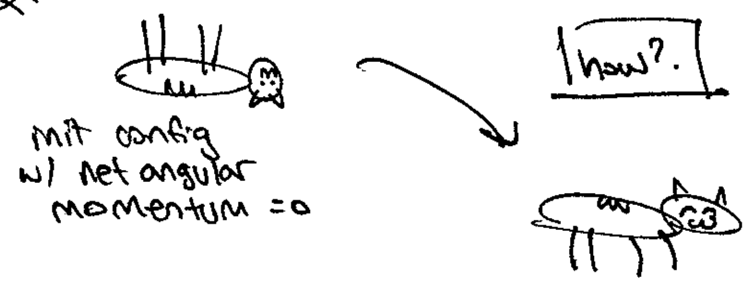
<u>remark</u>: $R \in$ LIE GROUP $SO(3)$

$\omega \in$ LIE ALGEBRA $so(3)$ OR $\mathcal{L}(SO(3))$

<u>the Falling Cat</u> : PART 1

↳ brings together some nice ideas that are deeply embedded in physics

Q:

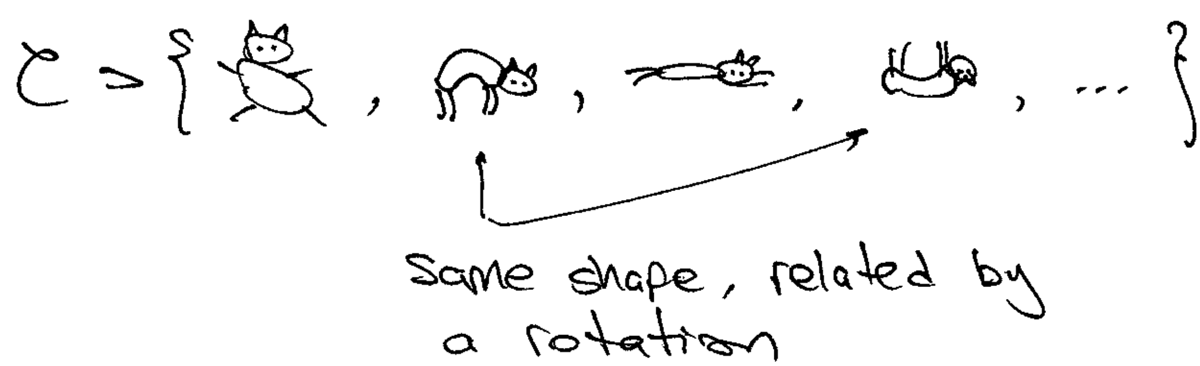


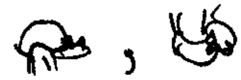
in fact, easy enough to imagine physically what happens — rotation of a pole along its axis costs very little angular momentum.

BUT WE CAN USE THIS TO EXPLORE OUR MACHINERY.

<u>main idea</u> : deformable bodies.

configuration space $\mathcal{C}$



In some sense these are the same: 

SO DEFINE A <u>SHAPE SPACE</u> $\tilde{\mathcal{C}} \subset \mathcal{C}$

$\tilde{\mathcal{C}} = \{$  $, \ldots \}$

<u>RELATION</u>: $\tilde{\mathcal{C}} = \mathcal{C} / SO(3)$

SHAPE SPACE

"MOD OUT" BY ROTATIONS

CONFIG SPACE

 , etc.

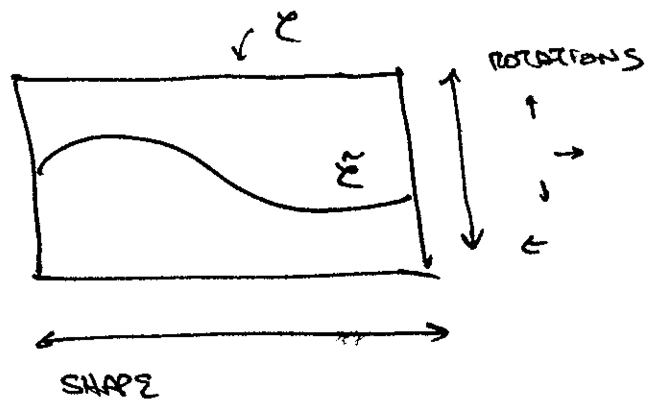
THESE DESCRIBE THE <u>SAME</u> SHAPE ; <u>one</u> element of $\tilde{\mathcal{C}}$. IN THE DIAGRAM ABOVE, WE HAVE CHOSEN A <u>REPRESENTATIVE</u> FOR EACH SHAPE.

consider the cat as a set of <u>non-rigid</u> points $\underline{r}_i$ w/ $\underbrace{|\underline{r}_i - \underline{r}_j| \neq \text{constant.}}_{\text{shape deformations allowed}}$

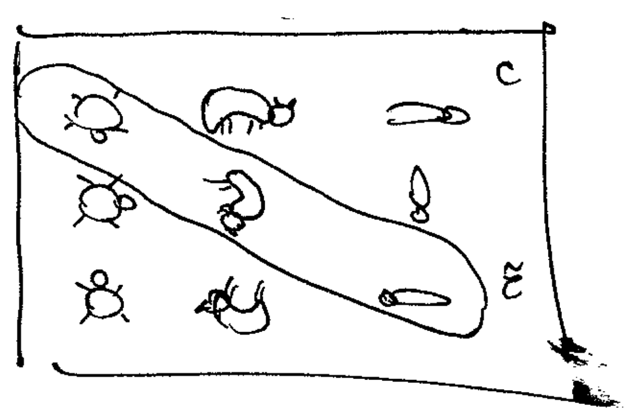
PICK A SET OF REFERENCE/REPRESENTATIVE CONFIGS FOR EACH SHAPE: $\{\tilde{\underline{r}}_i\}$ s.t. $\exists$ rotation R $\forall$ $\underline{r}_i$ s.t. $\underline{r}_i = R\,\tilde{\underline{r}}_i$

GRAPHICALLY:

<u>mathematically</u>
└ section of a fiber bundle.



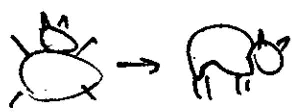
eg:



Want to understand: HOW TO ROTATE BY CHANGING SHAPE WHILE FIXING ANGULAR MOMENTUM $L = 0$.

conceptual issue: meaning of rotation b/w'n objects of different shapes?

Better:  GOES THROUGH A SEQUENCE OF SHAPES BEFORE RETURNING TO INITIAL SHAPE. ASK: HAS THERE BEEN A NET ROTATION.



has there been a rotation? (not well defined))

has there been a rotation?

IN ~~BODY~~ SPACE FRAME & ignoring gravity

$$\underline{r}_i(t) = R(t)_{ij}\, \underline{\tilde{r}}_j(t)$$

true config

connects REPRESENTATIVE TO TRUE CONFIG

REPRESENTATIVE (same shape, may have diff orientation)

CONNECTIVE ANGULAR VELOCITY

$$\Omega = R^{-1}\dot{R}$$

$\neq 0$ because object changes shape

COORDINATES ON SHAPE SPACE:

$x^A \qquad A = 1, \dots, 3N$ ← N-PARTICLE "CAT" s.t. $\{x^A\} \leftrightarrow$ SHAPE

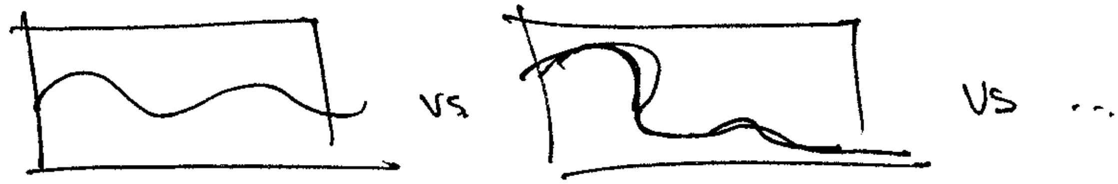
instead of $\tilde{r}_1 = (x_1, x_2, x_3)$
$\tilde{r}_2 = (x_4, x_5, x_6)$
$\vdots$

Then: $\Omega = \underbrace{\Omega_A(x)} \dot{x}^A(t)$ by linearity in $\frac{d}{dt}$

3x3 MATRIX OF ANGULAR VELOCITY IF SHAPE CHANGES s.t. $x^A \to x^A + \delta x^A$ TIME-INDEP.

contains all time dep. (PATH IN SHAPE SPACE)

Ambiguity: $\tilde{r}_i \in \tilde{\mathcal{E}}$ WAS CHOSEN ARBITRARILY:



COULD HAVE PICKED DIFFERENT REFERENCE ORIENTATION:

$$\tilde{r}_i \to S(x^A)\,\tilde{r}_i$$

OLD REF

ROTATION TO NEW REF.
DEPENDS ON THE SHAPE, x_A

then: $\underline{r}_i(t) = R_{ij}(t)\,\underline{\tilde{r}}_j(t)$

$$\downarrow$$

$$\underline{r}_i(t) = \left(R(t)\,S^{-1}(x_A)\right)_{ij}\,\underline{\tilde{r}}_j(t)$$

$$\Omega = R^{-1}\dot{R} = \Omega_A(x)\,\dot{x}^A$$

$$\downarrow$$

$$\left(S\Omega_A S^{-1} + S\frac{\partial S^{-1}}{\partial x^A}\right)$$

$$\left(\begin{array}{ll} \text{cf. } \Omega \sim A & S \sim e^{i\lambda(x)} \\ A \to & A + \partial\lambda \end{array}\right)$$

$\Omega \sim$ GAUGE POTENTIAL, CONNECTION ON FIBER BUNDLE