P661 Homework # 2 9/02/04

Due Tuesday 9/14/04

Part 1

(1) To check that $T = 1/2\pi\alpha'$ is the tension (energy per unit length) of the string, consider, at $X_0 = t = 0$ and $\tau = 0$, a circle in the $X_1 - X_2$-plane:

$$X_1 = R\cos(2\pi\sigma), \qquad X_2 = R\sin(2\pi\sigma) \tag{1}$$

Show that, after restoring the τ-dependence, it satisfies the wave equation

$$(\partial_\tau^2 - \partial_\sigma^2)X_\mu = 0$$

and the constraint equation $\partial_\tau X \cdot \partial_\sigma X = 0$.

The other constraint $\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X = 0$ is also satisfied if $X_0 = 2\pi R\tau$.

With $P_0 = T\partial_\tau X_0 = 2\pi RT$, the energy is $p_0 = \int_0^1 d\sigma P_0 = 2\pi RT$.

(2) Derive the Virasoro algebra. Ignore the constant term (the central charge) on the RHS.

Part 2 :

(3) Go over the light-cone quantization in Ch. 1.

(4) To check the ζ-function regularization, one may introduce a cutoff and then remove the divergent term via a world-sheet cosmological counter term. Consider

$$F(s) = \sum_{k=1}^{\infty} k^{-s} \tag{2}$$

for $s = -1$. Let $f(\epsilon) = \sum_{k=1}^{\infty} e^{-\epsilon k}$, so

$$F(-1) = -\lim_{\epsilon \to 0} \frac{\partial f(\epsilon)}{\partial \epsilon} \tag{3}$$

Compare with Eq.(1.3.34) in Ch.1.

Comment: In general, consider the Riemann zeta-function

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$$

For example, $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$. For negative s, we have

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}$$

where B_{2n} is the Bernouli number. Note that $B_2 = 1/6$.

(5) Derive the Virasoro algebra with the central charge.

You may find the central charge term either by brute force (normal ordering), or by a more streamlined approach explained in p80 in GSW I.

(6) Let us go back to Problem (1) above. Decompose the solution to its left-moving and right-moving parts, so (up to a sign and normalization)

$$\mathbf{X}_R(\sigma^-) \simeq \frac{1}{2}(\sin(2\pi\sigma^-), -\cos(2\pi\sigma^-), 0, 0, ...)$$

If one wants to add a component of a higher harmonic to this lowest harmonic solution, show that (for $1 > b \geq 0$)

$$\mathbf{X}_R(\sigma^-) = \frac{1}{2}\Big((1-b)\sin(2\pi\sigma^-) + \frac{b}{3}\sin(6\pi\sigma^-), -(1-b)\cos(2\pi\sigma^-) - \frac{b}{3}\cos(6\pi\sigma^-),$$
$$-2\sqrt{b(1-b)}\cos(2\pi\sigma^-), 0, ...\Big) \quad (4)$$

is the only solution. That is, the constraint only allows the inclusion of the 3rd harmonic (not 2nd, 4th, 5th, 6th etc.).